ApexMath

Lesson 7: Ellipses

Notes on Ellipses

- An ellipse is the set of points where the sum of distances from two fixed points (foci) is constant.
- **Standard forms** (centered at (h, k)):

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{(horizontal major axis)}$$

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1 \quad \text{(vertical major axis)}$$

  where $a^2 > b^2$.
- Key parts:
  - $a =$ semi-major axis
  - $b =$ semi-minor axis
  - $c =$ distance from center to each focus, $c^2 = a^2 - b^2$
  - eccentricity $e = \dfrac{c}{a}$
- **Directrices**:

$$x = h \pm \frac{a}{e} \quad \text{(horizontal ellipse)}, \qquad y = k \pm \frac{a}{e} \quad \text{(vertical ellipse)}.$$

Pro Tips

- The larger denominator indicates the major axis.
- Always use $c^2 = a^2 - b^2$ to find foci.
- Use a and c to compute eccentricity and directrices.
- Directrices lie outside the ellipse at distance a/e from the center.

Deep Walkthrough Example

Example A. Analyze the ellipse:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Put it into standard form and identify: the center, a, b, c, vertices, co-vertices, foci, eccentricity, and directrices.

Step 1. Identify a^2 and b^2:

$$a^2 = 25 \Rightarrow a = 5 \qquad b^2 = 9 \Rightarrow b = 3$$

Step 2. Orientation:

Horizontal major axis (since the larger denominator is under x^2).

Step 3. Center, vertices, co-vertices:

$$\text{Center } (0,0)$$

$$\text{Vertices } (\pm 5, 0)$$

$$\text{Co-vertices } (0, \pm 3)$$

Step 4. Foci:

$$c^2 = a^2 - b^2 = 25 - 9 = 16 \quad \Rightarrow \quad c = 4$$

Foci: $(\pm 4, 0)$.

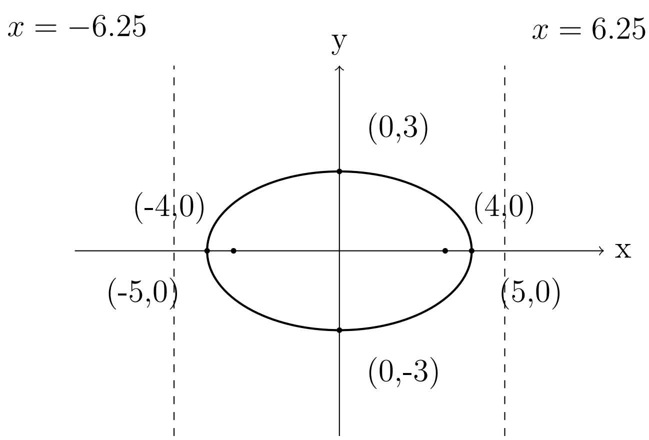
Step 5. Eccentricity:

$$e = \frac{c}{a} = \frac{4}{5} = 0.8$$

Step 6. Directrices:

$$x = \pm\frac{a}{e} = \pm\frac{5}{0.8} = \pm 6.25$$

Step 7. Graph:



$$\boxed{Center(0,0)\text{,a=5,b=3,c=4,e=0.8,}\,directrices\text{x=}\pm6.25}$$

Common Mistakes

- Mixing up a and b: always identify which denominator is larger.
- Forgetting to take the square root when computing c.
- Confusing orientation for foci and directrices.
- Neglecting exact versus decimal forms.

Worksheet Problems

1. Given $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$, put it into standard form and identify the center, a, b, c, vertices, co-vertices, foci, eccentricity, and directrices.
2. Given $9x^2 + 16y^2 = 144$, put it into standard form and identify the ellipse's properties.
3. Given $\dfrac{(y-2)^2}{25} + \dfrac{(x+1)^2}{9} = 1$, identify the center, a, b, c, vertices, co-vertices, foci, eccentricity, and directrices.
4. Write the equation of the ellipse with center $(0, 0)$, $a = 6$, $b = 2$, and a horizontal major axis. Then identify c, eccentricity, and directrices.
5. Write the equation of the ellipse with center $(0, 0)$, foci $(0, \pm 5)$, vertices $(0, \pm 13)$. Then identify a, b, c, eccentricity, and directrices.

Answer Key

1. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

Step 1. Identify a^2 and b^2:

$$a^2 = 16 \Rightarrow a = 4, \quad b^2 = 4 \Rightarrow b = 2$$

Step 2. Orientation: horizontal major axis.

Step 3. Center, vertices, co-vertices:

$$\text{Center } (0, 0)$$

$$\text{Vertices } (\pm 4, 0)$$

$$\text{Co-vertices } (0, \pm 2)$$

Step 4. Foci:

$$c^2 = a^2 - b^2 = 16 - 4 = 12 \quad \Rightarrow \quad c = 2\sqrt{3} \approx 3.464$$

Foci: $(\pm 2\sqrt{3}, 0)$

Step 5. Eccentricity:

$$e = \frac{c}{a} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \approx 0.866$$

Step 6. Directrices:

$$x = \pm\frac{a}{e} = \pm\frac{4}{\sqrt{3}/2} = \pm\frac{8}{\sqrt{3}} \approx \pm 4.618$$

$$\boxed{Center\text{(0,0),a=4,b=2,c=}2\sqrt{3} \approx 3.464,\ e = \tfrac{\sqrt{3}}{2} \approx 0.866,\ x = \pm\tfrac{8}{\sqrt{3}} \approx \pm 4.618}$$

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2. $9x^2 + 16y^2 = 144$

Step 1. Standard form:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Step 2. Identify a^2 and b^2:

$$a^2 = 16 \Rightarrow a = 4, \quad b^2 = 9 \Rightarrow b = 3$$

Step 3. Orientation: horizontal major axis.

Step 4. Center, vertices, co-vertices:

$$\text{Center } (0,0)$$

$$\text{Vertices } (\pm 4, 0)$$

$$\text{Co-vertices } (0, \pm 3)$$

Step 5. Foci:

$$c^2 = a^2 - b^2 = 16 - 9 = 7 \quad \Rightarrow \quad c = \sqrt{7} \approx 2.646$$

Foci: $(\pm\sqrt{7}, 0)$

Step 6. Eccentricity:

$$e = \frac{c}{a} = \frac{\sqrt{7}}{4} \approx 0.661$$

Step 7. Directrices:

$$x = \pm\frac{a}{e} = \pm\frac{4}{\sqrt{7}/4} = \pm\frac{16}{\sqrt{7}} \approx \pm 6.053$$

$$\boxed{Center(0,0)\text{,a=4,b=3,c=}\sqrt{7} \approx 2.646,\ e \approx 0.661,\ x \approx \pm 6.053}$$

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3. $\dfrac{(y-2)^2}{25} + \dfrac{(x+1)^2}{9} = 1$

Step 1. Identify a^2 and b^2:

$$a^2 = 25 \Rightarrow a = 5, \quad b^2 = 9 \Rightarrow b = 3$$

Step 2. Orientation: vertical major axis.

Step 3. Center, vertices, co-vertices:

$$\text{Center } (-1, 2)$$

$$\text{Vertices } (-1, 2 \pm 5) = (-1, 7), (-1, -3)$$

$$\text{Co-vertices } (-1 \pm 3, 2) = (2, 2), (-4, 2)$$

Step 4. Foci:

$$c^2 = a^2 - b^2 = 25 - 9 = 16 \quad \Rightarrow \quad c = 4$$

Foci: $(-1, 2 \pm 4) = (-1, 6), (-1, -2)$

Step 5. Eccentricity:

$$e = \frac{c}{a} = \frac{4}{5} = 0.8$$

Step 6. Directrices:

$$y = k \pm \frac{a}{e} = 2 \pm \frac{5}{0.8} = 2 \pm 6.25$$

$$\boxed{\textit{Center}\text{(-1,2),a=5,b=3,c=4,e=0.8, }\textit{directrices}\text{y=2±6.25}}$$

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4. Center $(0, 0)$, $a = 6$, $b = 2$

Step 1. Equation:

$$\frac{x^2}{36} + \frac{y^2}{4} = 1$$

Step 2. Orientation: horizontal major axis.

Step 3. Vertices and co-vertices:

$$\text{Vertices } (\pm 6, 0), \quad \text{Co-vertices } (0, \pm 2)$$

Step 4. Foci:

$$c^2 = 36 - 4 = 32 \quad \Rightarrow \quad c = 4\sqrt{2} \approx 5.657$$

Foci: $(\pm 4\sqrt{2} \approx 5.657, 0)$

Step 5. Eccentricity:

$$e = \frac{c}{a} = \frac{4\sqrt{2}}{6} \approx 0.943$$

Step 6. Directrices:

$$x = \pm\frac{a}{e} = \pm\frac{6}{0.943} \approx \pm 6.364$$

$$\boxed{\frac{x^2}{36} + \frac{y^2}{4} = 1,\ c \approx 5.657,\ e \approx 0.943,\ x \approx \pm 6.364}$$

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5. Center $(0,0)$, foci $(0,\pm 5)$, vertices $(0,\pm 13)$

Step 1. Identify a and c:

$$a = 13, \quad c = 5$$

Step 2. Find b:

$$b^2 = a^2 - c^2 = 169 - 25 = 144 \quad \Rightarrow \quad b = 12$$

Step 3. Orientation: vertical major axis.

Step 4. Equation:

$$\frac{y^2}{169} + \frac{x^2}{144} = 1$$

Step 5. Eccentricity:

$$e = \frac{c}{a} = \frac{5}{13} \approx 0.385$$

Step 6. Directrices:

$$y = \pm \frac{a}{e} = \pm \frac{13}{0.385} \approx \pm 33.766$$

$$\boxed{\frac{y^2}{169} + \frac{x^2}{144} = 1,\ a = 13,\ b = 12,\ c = 5,\ e \approx 0.385,\ y \approx \pm 33.766}$$

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