Sequences & Series
Lesson 3: Geometric Series & Sums
ApexMath

Notes

A **geometric sequence** has a constant ratio r between terms.

$$a_n = a_1 \cdot r^{\,n-1}$$

Geometric Series (finite sum): If you add the first n terms of a geometric sequence:

$$S_n = a_1 \cdot \frac{1 - r^n}{1 - r}, \quad r \neq 1$$

Infinite Series: If $|r| < 1$, the infinite sum converges to

$$S_\infty = \frac{a_1}{1 - r}$$

Variables: - a_1: first term - r: common ratio - n: number of terms - a_n: the n^{th} term - S_n: sum of first n terms

—

Deep Walkthrough Examples

Example A. Finite sum
Find the sum of the first 8 terms of $2, 6, 18, \ldots$.

Step 1. Identify: $a_1 = 2$, $r = \frac{6}{2} = 3$, $n = 8$.

Step 2. Formula:

$$S_n = a_1 \cdot \frac{1 - r^n}{1 - r}$$

Step 3. Substitute:

$$S_8 = 2 \cdot \frac{1 - 3^8}{1 - 3}$$

$$S_8 = 2 \cdot \frac{1 - 6561}{-2} = 2 \cdot \frac{-6560}{-2} = 6560$$

Final Answer: $\boxed{6560}$

—

Example B. Infinite sum
Find the infinite sum of $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$

Step 1. Identify: $a_1 = 5$, $r = \frac{1}{2}$, $|r| < 1$.

Step 2. Formula:

$$S_\infty = \frac{a_1}{1-r}$$

Step 3. Compute:

$$S_\infty = \frac{5}{1-\frac{1}{2}} = \frac{5}{\frac{1}{2}} = 10$$

Final Answer: $\boxed{10}$

—

Example C. Solve for number of terms
How many terms are needed in $1 + 2 + 4 + 8 + \ldots$ to reach sum $S_n = 511$?

Step 1. Identify: $a_1 = 1$, $r = 2$, $S_n = 511$.

Step 2. Formula:

$$S_n = \frac{1-r^n}{1-r}$$

Step 3. Substitute:

$$511 = \frac{1-2^n}{1-2} = (2^n - 1)$$

Step 4. Solve:

$$511 = 2^n - 1 \quad \Rightarrow \quad 2^n = 512$$

$$n = 9$$

Final Answer: $\boxed{9 \text{ terms}}$

—

Common Mistakes to Avoid

- Using the arithmetic formula for a geometric series.
- Forgetting that $|r| < 1$ is required for the infinite sum.

- Plugging r^n with $n-1$ by mistake.
- Mixing up a_n (term) and S_n (sum).

—

Worksheet Problems

1. Find the sum of the first 6 terms of $3, 9, 27, \ldots$

2. Find S_{10} for $a_1 = 4$, $r = 2$.

3. A geometric sequence has $a_1 = 7$, $r = \frac{1}{3}$. Find S_∞.

4. In a sequence $a_1 = 5$, $r = 3$, find how many terms are needed for $S_n = 1210$.

5. The sum of the first 12 terms of $10, 5, 2.5, \ldots$.

6. Find a_{12} and S_{12} for $a_1 = 2$, $r = \frac{1}{2}$.

7. A ball is dropped from 20 m, bouncing to $\frac{3}{5}$ of its previous height. Write the infinite sum of its bounce heights.

8. For $a_1 = 1$, $r = -\frac{1}{2}$, find S_8.

9. $a_1 = 100$, $r = 0.9$. Find S_{15}.

10. A company pays \$1000 in year 1, then doubles payment each year. How much is paid in total by the end of year 5?

11. The sequence $4, 8, 16, 32, \ldots$. Find the infinite sum. (Trick Question!)

—

Detailed Answer Key

1. $a_1 = 3, r = 3, n = 6$

$$S_6 = 3 \cdot \frac{1-3^6}{1-3} = 3 \cdot \frac{1-729}{-2} = 3 \cdot 364 = 1092$$

Final Answer: $\boxed{1092}$

---

2. $a_1 = 4, r = 2, n = 10$

$$S_{10} = 4 \cdot \frac{1 - 2^{10}}{1 - 2} = 4(1023) = 4092$$

Final Answer: $\boxed{4092}$

---

3. $a_1 = 7, r = \frac{1}{3}$

$$S_\infty = \frac{7}{1 - \frac{1}{3}} = \frac{7}{\frac{2}{3}} = \frac{21}{2} = 10.5$$

Final Answer: $\boxed{10.5}$

---

4. $a_1 = 5, r = 3, S_n = 1210$

$$S_n = \frac{5(1 - 3^n)}{1 - 3} = \frac{5(3^n - 1)}{2}$$

Set equal:

$$1210 = \frac{5(3^n - 1)}{2} \quad \Rightarrow \quad 2420 = 5(3^n - 1)$$
$$484 = 3^n - 1 \quad \Rightarrow \quad 3^n = 485$$

$n = \log_3(485) \approx 6.136 \to$ Not integer.

Closest integer: $n = 5,\ S_5 = 605$; $n = 6,\ S_6 = 1820$.

Answer: No exact integer solution. Between 5 and 6 terms.

---

5. $a_1 = 10, r = \frac{1}{2}, n = 12$

$$S_{12} = 10 \cdot \frac{1 - (\frac{1}{2})^{12}}{1 - \frac{1}{2}} = 20\left(1 - \tfrac{1}{4096}\right) \approx 19.995$$

Final Answer: $\boxed{19.995}$

---

6. $a_1 = 2, r = \frac{1}{2}, n = 12$

$$a_{12} = 2 \cdot \left(\tfrac{1}{2}\right)^{11} = \tfrac{2}{2048} = \tfrac{1}{1024}$$

$$S_{12} = 2 \cdot \frac{1 - (\frac{1}{2})^{12}}{1 - \frac{1}{2}} = 4\left(1 - \tfrac{1}{4096}\right) = 3.9998$$

Final Answer: $\boxed{\mathrm{a}_{12} = \tfrac{1}{1024},\ S_{12} \approx 3.9998;}$

---

7. Bounce heights: $a_1 = 20 \cdot \frac{3}{5} = 12$, $r = \frac{3}{5}$

$$S_\infty = \frac{12}{1-\frac{3}{5}} = \frac{12}{\frac{2}{5}} = 30$$

Final Answer: $\boxed{30 m total bounce height}$

---

8. $a_1 = 1, r = -\frac{1}{2}, n = 8$

$$S_8 = 1 \cdot \frac{1-(-\frac{1}{2})^8}{1-(-\frac{1}{2})} = \frac{1-(1/256)}{1.5}$$

$$= \frac{255/256}{1.5} = \frac{255}{384} = 0.664$$

Final Answer: $\boxed{0.664}$

---

9. $a_1 = 100, r = 0.9, n = 15$

$$S_{15} = 100 \cdot \frac{1-(0.9)^{15}}{1-0.9} = 1000(1-0.2059) = 794.1$$

Final Answer: $\boxed{794.1}$

---

10. Payments: $a_1 = 1000, r = 2, n = 5$

$$S_5 = 1000 \cdot \frac{1-2^5}{1-2} = 1000(31) = 31000$$

Final Answer: $\boxed{\$31000}$

---

11. Trick question: $a_1 = 4,\ r = 2$

Step 1. Formula for infinite series:

$$S_\infty = \frac{a_1}{1-r}$$

Step 2. Substitute:

$$S_\infty = \frac{4}{1-2} = \frac{4}{-1} = -4$$

Step 3. But this result is meaningless because the formula only works if $|r| < 1$. Here $r = 2$, so $|r| > 1$. That means each term grows larger:

$$4, 8, 16, 32, 64, \cdots \to \infty$$

Step 4. Since the terms do not shrink toward zero, the series cannot settle to a finite value.

Final Answer: $\boxed{\text{Diverges (no finite sum)}}$

—

Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

End of Lesson • ApexMath