ApexMath

Lesson 9 — Hyperbolas

Learning goals

By the end of this lesson you should be able to:

- Recognize and write the two standard forms of a hyperbola.
- Compute a, b, c, vertices, foci, asymptotes, eccentricity e, and directrices.
- Convert a general conic equation into the standard hyperbola form by completing the square.
- Use the asymptote slope trick to sketch quickly and check answers for consistency.

Notes — taught clearly

A hyperbola consists of two separate branches. It can open left/right (horizontal transverse axis) or up/down (vertical transverse axis). Important forms:

Horizontal transverse axis (opens left/right):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Vertical transverse axis (opens up/down):

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Key relationships and intuition: Distance relation between a, b, c:

$$c^2 = a^2 + b^2$$

Eccentricity e definition and range:

$$e = \frac{c}{a} \quad \text{and for hyperbolas } e > 1$$

Directrices (intuition):

Each directrix is a line $\frac{a}{e}$ away from the center along the transverse axis

A few teaching pointers for struggling students:

- Always identify the center (h, k) from the shifts in the squared terms first — this avoids sign errors later.
- The denominator under the positive squared term is a^2. Write down a^2, b^2 immediately and take square roots carefully.
- Use exact expressions (e.g. fractions and radicals) first, then approximate decimals at the end.

Pro trick (asymptote slope, quick intuition)

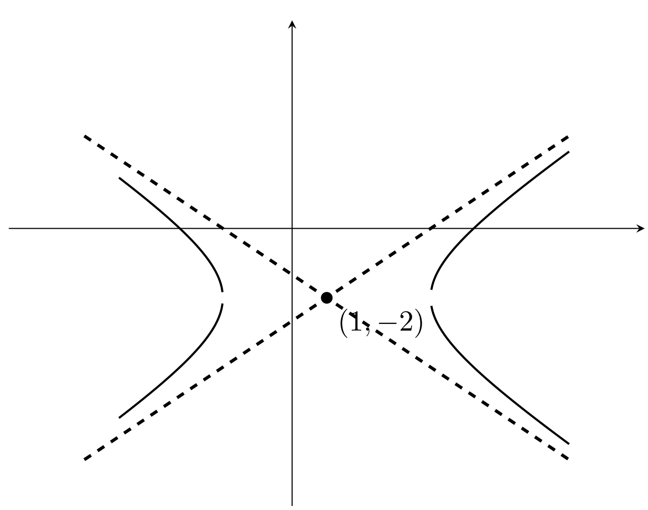
At large distances the "1" becomes negligible and the hyperbola behaves like two lines. The slope uses the denominator under the other variable:

$$\text{horizontal: } y - k = \pm\frac{b}{a}(x - h), \qquad \text{vertical: } y - k = \pm\frac{a}{b}(x - h).$$

Clean graph example (only hyperbola + asymptotes)

Example hyperbola to use as a model:

$$\frac{(x-1)^2}{9} - \frac{(y+2)^2}{4} = 1$$



Deep Walkthroughs

Example 1 (standard form — horizontal)

Read center directly from shifts:

$$(h, k) = (1, -2)$$

Denominators give a^2 and b^2:

$$a^2 = 9 \Rightarrow a = 3, \quad b^2 = 4 \Rightarrow b = 2$$

Compute c using $c^2 = a^2 + b^2$:

$$c^2 = 9 + 4 = 13 \Rightarrow c = \sqrt{13} \approx 3.606$$

Vertices (move a along transverse axis):

$$(h \pm a, k) = (1 \pm 3, -2) = (-2, -2), (4, -2)$$

Foci (move c along transverse axis):

$$(h \pm c, k) = (1 \pm \sqrt{13}, -2)$$

Asymptotes (horizontal slopes $\pm b/a$):

$$y + 2 = \pm\frac{2}{3}(x - 1)$$

Eccentricity $e = c/a$:

$$e = \frac{\sqrt{13}}{3} \approx 1.202$$

Directrices (exact then approximate):

$$x = h \pm \frac{a}{e} = 1 \pm \frac{a^2}{c} = 1 \pm \frac{9}{\sqrt{13}} \approx -1.498,\ 3.498$$

Summary (Example 1)
Center: $(1, -2)$.
$a = 3,\ b = 2,\ c = \sqrt{13}$.
Vertices: $(-2, -2), (4, -2)$.
Foci: $(1 \pm \sqrt{13}, -2)$.
Asymptotes: $y + 2 = \pm\frac{2}{3}(x - 1)$.
Eccentricity: $e = \frac{\sqrt{13}}{3} \approx 1.202$.
Directrices: $x = 1 \pm \frac{9}{\sqrt{13}} \approx -1.498,\ 3.498$.

Example 2 (general form → standard form)

Start with the equation:

$$4y^2 - 9x^2 + 16y + 36x - 36 = 0$$

Group y terms and x terms for completing the square:

$$4(y^2 + 4y) - 9(x^2 - 4x) - 36 = 0$$

Complete squares inside groups and simplify constants:

$$4(y + 2)^2 - 9(x - 2)^2 - 16 = 0$$

Move constant and divide to obtain standard form:

$$\frac{(y + 2)^2}{4} - \frac{(x - 2)^2}{16/9} = 1$$

Read off center and parameters:

$$(h,k)=(2,-2),\quad a^2=4\Rightarrow a=2,\quad b^2=\tfrac{16}{9}\Rightarrow b=\tfrac{4}{3}$$

Compute c:

$$c^2=a^2+b^2=4+\tfrac{16}{9}=\tfrac{52}{9}\Rightarrow c=\tfrac{\sqrt{52}}{3}\approx 2.404$$

Vertices and foci:

$$\text{Vertices } (2,-2\pm 2);\quad \text{Foci } (2,-2\pm 2.404)$$

Asymptotes (vertical transverse $\to$ slope $\pm a/b$):

$$y+2=\pm\tfrac{3}{2}(x-2)$$

Eccentricity:

$$e=\frac{c}{a}\approx 1.202$$

Summary (Example 2)
Center: $(2,-2)$.
$a=2,\ b=\tfrac{4}{3},\ c=\tfrac{\sqrt{52}}{3}\approx 2.404.$
Vertices: $(2,-4),(2,0)$.
Foci: $(2,-4.404),(2,0.404)$.
Asymptotes: $y+2=\pm\tfrac{3}{2}(x-2)$.
Eccentricity: $e\approx 1.202$.

Common mistakes (how to spot and fix them)

- **Mixing ellipse vs hyperbola formulas.** Fix: hyperbola uses $c^2=a^2+b^2$. If you accidentally get $c^2=a^2-b^2$, you have the ellipse formula and the rest will be wrong.
- **Choosing the wrong a^2.** Fix: the denominator under the positive squared term is a^2. Always find the positive term first.
- **Forgetting center shifts.** Fix: write (h,k) down and add it back to every vertex, focus, and asymptote equation.
- **Arithmetic errors when completing squares.** Fix: write what you add/subtract explicitly and move constants to the other side before dividing to make RHS $=1$.

Worksheet (do each step and show work)

1. $\dfrac{(x+2)^2}{16} - \dfrac{(y-1)^2}{9} = 1.$
2. $4y^2 - 9x^2 + 16y + 36x - 36 = 0.$
3. Center $(0,0)$, vertices $(0,\pm 5)$, $e = \dfrac{13}{5}$.
4. Foci $(\pm 10, 0)$, vertices $(\pm 6, 0)$.
5. $9x^2 - 24x - 16y^2 + 64y + 31 = 0.$

Detailed Answer Key

Problem 1

Start: read shifts from $(x+2)^2$ and $(y-1)^2$:

$$(h,k) = (-2,1)$$

Denominators give a^2 and b^2:

$$a^2 = 16 \Rightarrow a = 4, \quad b^2 = 9 \Rightarrow b = 3$$

Compute c:

$$c^2 = a^2 + b^2 = 16 + 9 = 25 \Rightarrow c = 5$$

Vertices (move a from center along transverse axis):

$$(-2 \pm 4, 1) = (-6,1), (2,1)$$

Foci (move c from center):

$$(-2 \pm 5, 1) = (-7,1), (3,1)$$

Asymptotes (horizontal $\pm b/a$):

$$y - 1 = \pm\frac{3}{4}(x + 2)$$

Eccentricity:

$$e = \frac{c}{a} = \frac{5}{4} = 1.25$$

Problem 1
Center $(-2, 1)$; $a = 4,\ b = 3,\ c = 5.$
Vertices: $(-6, 1), (2, 1)$. Foci: $(-7, 1), (3, 1)$.
Asymptotes: $y - 1 = \pm\frac{3}{4}(x + 2)$. $e = 1.25$.

Problem 2

Start: group and complete squares:

$$4(y^2 + 4y) - 9(x^2 - 4x) - 36 = 0$$

Complete squares and simplify:

$$4(y + 2)^2 - 9(x - 2)^2 - 16 = 0$$

Divide to get standard form:

$$\frac{(y + 2)^2}{4} - \frac{(x - 2)^2}{16/9} = 1$$

Center and parameters:

$$(h, k) = (2, -2),\ a^2 = 4 \Rightarrow a = 2,\ b^2 = \tfrac{16}{9} \Rightarrow b = \tfrac{4}{3}$$

Compute c:

$$c^2 = 4 + \tfrac{16}{9} = \tfrac{52}{9} \Rightarrow c = \tfrac{\sqrt{52}}{3} \approx 2.404$$

Vertices and foci:

$$\text{Vertices } (2, -2 \pm 2); \quad \text{Foci } (2, -2 \pm 2.404)$$

Asymptotes (vertical transverse, slope $\pm a/b$):

$$y + 2 = \pm \tfrac{3}{2}(x - 2)$$

Eccentricity:

$$e = \frac{c}{a} \approx 1.202$$

Problem 2
Center $(2, -2)$; $a = 2,\ b = \tfrac{4}{3},\ c = \tfrac{\sqrt{52}}{3} \approx 2.404$.
Vertices: $(2, -4), (2, 0)$. Foci: $(2, -4.404), (2, 0.404)$.
Asymptotes: $y + 2 = \pm \tfrac{3}{2}(x - 2)$. $e \approx 1.202$.

Problem 3

Given vertices $(0, \pm 5)$ so $a = 5$, center $(0, 0)$:

Eccentricity given $e = \tfrac{13}{5}$ so compute c:

$$c = ea = \frac{13}{5} \cdot 5 = 13$$

Compute b from $c^2 = a^2 + b^2$:

$$b^2 = c^2 - a^2 = 169 - 25 = 144 \Rightarrow b = 12$$

Standard equation (vertical transverse):

$$\frac{y^2}{25} - \frac{x^2}{144} = 1$$

Asymptotes (vertical):

$$y = \pm\frac{a}{b}x = \pm\frac{5}{12}x$$

Directrices (vertical case):

$$y = \pm\frac{a}{e} = \pm\frac{5}{13/5} = \pm\frac{25}{13} \approx \pm 1.923$$

Problem 3

Equation: $\dfrac{y^2}{25} - \dfrac{x^2}{144} = 1$. Center $(0, 0)$.

$a = 5,\ b = 12,\ c = 13$. Vertices: $(0, \pm 5)$. Foci: $(0, \pm 13)$.

Asymptotes: $y = \pm\frac{5}{12}x$. Directrices: $y = \pm\frac{25}{13} \approx \pm 1.923$. $e = 2.6$.

Problem 4

Given vertices $(\pm 6, 0) \Rightarrow a = 6$ and foci $(\pm 10, 0) \Rightarrow c = 10$:

Compute b:

$$b^2 = c^2 - a^2 = 100 - 36 = 64 \Rightarrow b = 8$$

Standard equation (horizontal):

$$\frac{x^2}{36} - \frac{y^2}{64} = 1$$

Asymptotes (horizontal):

$$y = \pm\frac{b}{a}x = \pm\frac{8}{6}x = \pm\frac{4}{3}x$$

Directrices:

$$x = \pm\frac{a}{e} \text{ where } e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3} \Rightarrow x = \pm 3.600$$

Eccentricity:

$$e = \frac{10}{6} \approx 1.667$$

Problem 4
Equation: $\frac{x^2}{36} - \frac{y^2}{64} = 1$. Center $(0, 0)$.
$a = 6,\ b = 8,\ c = 10$. Vertices: $(\pm 6, 0)$. Foci: $(\pm 10, 0)$.
Asymptotes: $y = \pm\frac{4}{3}x$. Directrices: $x = \pm 3.600$. $e \approx 1.667$.

Problem 5

Start with the equation:

$$9x^2 - 24x - 16y^2 + 64y + 31 = 0$$

Group for completing squares:

$$9(x^2 - \tfrac{8}{3}x) - 16(y^2 - 4y) + 31 = 0$$

Complete the square carefully (show constants):

$$9\big(x - \tfrac{4}{3}\big)^2 - 16(y-2)^2 - 17 = 0$$

Move constant and divide to get standard form (horizontal):

$$9\big(x - \tfrac{4}{3}\big)^2 - 16(y-2)^2 = 17$$

Divide by 17:

$$\frac{(x - \frac{4}{3})^2}{17/9} - \frac{(y-2)^2}{17/16} = 1$$

So a^2 and b^2 are:

$$a^2 = \tfrac{17}{9}, \quad b^2 = \tfrac{17}{16}$$

Numeric values:

$$a = \frac{\sqrt{17}}{3} \approx 1.374, \quad b = \frac{\sqrt{17}}{4} \approx 1.031$$

Compute c:

$$c^2 = a^2 + b^2 = \tfrac{17}{9} + \tfrac{17}{16} = \tfrac{425}{144} \Rightarrow c = \tfrac{\sqrt{425}}{12} \approx 1.719$$

Asymptotes (horizontal case, slope $\pm b/a$):

$$y - 2 = \pm\frac{b}{a}(x - \tfrac{4}{3}) = \pm\frac{(\sqrt{17}/4)}{(\sqrt{17}/3)}(x - \tfrac{4}{3}) = \pm\frac{3}{4}(x - \tfrac{4}{3})$$

Eccentricity $e = c/a$ (exact then numeric):

$$e = \frac{c}{a} = \frac{\frac{\sqrt{425}}{12}}{\frac{\sqrt{17}}{3}} = \frac{5}{4} = 1.25$$

Directrices (exact formula $x = h \pm a/e$, numeric):

$$x = \tfrac{4}{3} \pm \frac{a}{e} \approx \tfrac{4}{3} \pm \frac{1.374}{1.25} \Rightarrow x \approx 2.432,\ 0.235$$

Problem 5

Standard form: $\dfrac{(x-\frac{4}{3})^2}{17/9} - \dfrac{(y-2)^2}{17/16} = 1$. Center: $(\frac{4}{3}, 2)$.
$a = \frac{\sqrt{17}}{3} \approx 1.374,\ b = \frac{\sqrt{17}}{4} \approx 1.031,\ c = \frac{\sqrt{425}}{12} \approx 1.719.$
Vertices: $(\frac{4}{3} \pm a,\ 2) \approx (-0.041, 2), (2.707, 2)$. Foci: $(\frac{4}{3} \pm c,\ 2) \approx (-0.386, 2), (3.053, 2)$.
Asymptotes: $y - 2 = \pm\frac{3}{4}(x - \frac{4}{3})$. Directrices: $x \approx 2.432,\ 0.235$. $e = 1.25$.

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End of Lesson • ApexMath
