Applications of Integrals — Answer Key
ApexMath

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1. The area under a curve $y = f(x)$ from $x = a$ to $x = b$ is given by:

This is the definition of a definite integral. The area is found by summing infinitely thin vertical slices of height $f(x)$ and width dx.

$$\boxed{\int_a^b f(x)\,dx}$$

2. Find the area under $y = \sqrt{x}$ from $x = 0$ to $x = 4$.

Step 1: Set up the integral.

$$\int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx$$

Step 2: Apply the power rule for integration. Recall that $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$. Here, $n = \frac{1}{2}$, so $n + 1 = \frac{3}{2}$.

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^4 = \left[\frac{2}{3}x^{3/2}\right]_0^4$$

Step 3: Evaluate at the bounds.

$$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2}$$

Step 4: Simplify. $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$.

$$= \frac{2}{3}(8) - 0 = \frac{16}{3}$$

$$\boxed{\frac{16}{3}}$$

3. The area between two curves $f(x)$ and $g(x)$ where $f(x) \geq g(x)$ on $[a, b]$ is:

Subtract the lower curve from the upper curve at every x-value, then integrate over the interval.

$$\boxed{\int_a^b \big[f(x) - g(x)\big]\,dx}$$

4. Find the area between $y = 2x$ and $y = x^2$ from $x = 0$ to $x = 2$.

Step 1: Determine which curve is on top. At $x = 1$: $2(1) = 2$ and $(1)^2 = 1$. Since $2 > 1$, the line $y = 2x$ is above $y = x^2$ on $[0, 2]$.

Step 2: Set up the integral.

$$\int_0^2 \left(2x - x^2\right) dx$$

Step 3: Integrate term by term.

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

Step 4: Evaluate at the bounds.

$$= \left(4 - \frac{8}{3}\right) - (0 - 0) = 4 - \frac{8}{3}$$

Step 5: Simplify. $4 = \frac{12}{3}$, so:

$$= \frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$

$$\boxed{\frac{4}{3}}$$

5. The disk method formula for rotating around the x-axis is:

Each thin vertical slice of the region becomes a disk with radius $f(x)$ and thickness dx. The volume of each disk is $\pi r^2 \, dx = \pi[f(x)]^2 \, dx$.

$$\boxed{V = \pi \int_a^b \left[f(x)\right]^2 dx}$$

6. Find the volume when $y = \sqrt{x}$ is revolved around the x-axis from $x = 0$ to $x = 4$.

Step 1: Identify the method and the radius. Use the disk method. The radius of each disk is $f(x) = \sqrt{x}$.

$$V = \pi \int_0^4 \left[\sqrt{x}\right]^2 dx$$

Step 2: Simplify the integrand. $\left[\sqrt{x}\right]^2 = x$.

$$V = \pi \int_0^4 x\,dx$$

Step 3: Integrate.

$$= \pi \left[\frac{x^2}{2}\right]_0^4$$

Step 4: Evaluate at the bounds.

$$= \pi \left(\frac{16}{2} - 0\right) = \pi(8) = 8\pi$$

$$\boxed{8\pi}$$

7. The washer method formula accounts for:

When a region between two curves is revolved, the solid has an outer wall and an inner hole. The washer method subtracts the volume of the inner disk (the hole) from the outer disk.

$$\boxed{\text{A solid with a hole in the center}}$$

8. Find the volume using the washer method for the region between $y = x$ and $y = x^2$ revolved around the x-axis from $x = 0$ to $x = 1$.

Step 1: Identify the outer and inner radii. On $[0, 1]$, check which is larger at $x = 0.5$: $y = 0.5$ vs. $y = 0.25$. So $y = x$ is the outer radius $R(x) = x$, and $y = x^2$ is the inner radius $r(x) = x^2$.

Step 2: Set up the washer formula.

$$V = \pi \int_0^1 \left(\left[R(x)\right]^2 - \left[r(x)\right]^2 \right) dx = \pi \int_0^1 \left(x^2 - x^4\right) dx$$

Step 3: Integrate term by term.

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

Step 4: Evaluate at the bounds.

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right) - \pi(0)$$

Step 5: Combine the fractions. $\frac{1}{3} - \frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}$

$$= \pi \cdot \frac{2}{15} = \frac{2\pi}{15}$$

$$\boxed{\frac{2\pi}{15}}$$

9. The shell method formula for rotating around the y-axis is:

Each thin vertical strip at position x becomes a cylindrical shell with radius x, height $f(x)$, and thickness dx. Its volume is $2\pi x \cdot f(x)\, dx$.

$$\boxed{V = 2\pi \int_a^b x \cdot f(x)\, dx}$$

10. Find the volume using the shell method when $y = x$ is revolved around the y-axis from $x = 0$ to $x = 2$.

Step 1: Identify the shell radius and height. The shell radius is x and the shell height is $f(x) = x$.

$$V = 2\pi \int_0^2 x \cdot x \, dx = 2\pi \int_0^2 x^2 \, dx$$

Step 2: Integrate.

$$= 2\pi \left[\frac{x^3}{3} \right]_0^2$$

Step 3: Evaluate at the bounds.

$$= 2\pi \left(\frac{8}{3} - 0 \right) = \frac{16\pi}{3}$$

$$\boxed{\frac{16\pi}{3}}$$

11. The arc length formula for $y = f(x)$ from $x = a$ to $x = b$ is:

Each tiny piece of the curve has length $\sqrt{(dx)^2 + (dy)^2}$. Factoring out dx gives $\sqrt{1 + \left(f'(x)\right)^2}\, dx$, which is then integrated.

$$\boxed{L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx}$$

12. Find the arc length of $y = x$ from $x = 0$ to $x = 1$.

Step 1: Find $f'(x)$. $f(x) = x$, so $f'(x) = 1$.

Step 2: Set up the arc length integral.

$$L = \int_0^1 \sqrt{1 + (1)^2}\, dx = \int_0^1 \sqrt{2}\, dx$$

Step 3: Integrate. $\sqrt{2}$ is a constant.

$$= \sqrt{2}\, [x]_0^1 = \sqrt{2}(1 - 0) = \sqrt{2}$$

$$\boxed{\sqrt{2}}$$

13. Find the arc length of $y = 2x^{3/2}$ from $x = 0$ to $x = 4$.

Step 1: Find $f'(x)$.

$$f'(x) = 2 \cdot \frac{3}{2}\, x^{1/2} = 3\sqrt{x}$$

Step 2: Compute $\left[f'(x)\right]^2$.

$$\left[f'(x)\right]^2 = \left(3\sqrt{x}\right)^2 = 9x$$

Step 3: Set up the arc length integral.

$$L = \int_0^4 \sqrt{1 + 9x}\, dx$$

Step 4: Use substitution. Let $u = 1 + 9x$, so $du = 9\, dx$, meaning $dx = \dfrac{du}{9}$.

When $x = 0$: $u = 1$. When $x = 4$: $u = 37$.

$$L = \int_1^{37} \sqrt{u} \cdot \frac{du}{9} = \frac{1}{9} \int_1^{37} u^{1/2}\, du$$

Step 5: Integrate.

$$= \frac{1}{9} \left[\frac{2}{3}\, u^{3/2}\right]_1^{37} = \frac{2}{27} \left[u^{3/2}\right]_1^{37}$$

Step 6: Evaluate at the bounds.

$$= \frac{2}{27}\left(37^{3/2} - 1^{3/2}\right) = \frac{2}{27}\left(37\sqrt{37} - 1\right) \approx 16.6$$

$$\boxed{\frac{2}{27}\left(37\sqrt{37} - 1\right) \approx 16.6}$$

14. To find the volume of a solid with known cross-sections perpendicular to the x-axis, use: At each x-value, the solid has a cross-section whose area is $A(x)$. Integrating those areas along x gives the total volume.

$$\boxed{V = \int_a^b A(x)\, dx, \text{ where } A(x) \text{ is the area of the cross-section}}$$

15. Find the volume of a solid with square cross-sections where the side length is $s(x) = x$, from $x = 0$ to $x = 3$.

Step 1: Find the area of each square cross-section. A square with side length s has area $A = s^2$. Here, $s(x) = x$, so:

$$A(x) = x^2$$

Step 2: Set up the volume integral.

$$V = \int_0^3 x^2\,dx$$

Step 3: Integrate.

$$= \left[\frac{x^3}{3}\right]_0^3$$

Step 4: Evaluate at the bounds.

$$= \frac{27}{3} - 0 = 9$$

$$\boxed{9}$$

16. Work done by a force $F(x)$ from $x = a$ to $x = b$ is:

Work is force times distance. When the force varies, we add up infinitely many tiny contributions $F(x)\,dx$ over the path.

$$\boxed{W = \int_a^b F(x)\,dx}$$

17. Find the work done against a spring force $F(x) = 3x$ from $x = 0$ to $x = 2$.

Step 1: Set up the work integral.

$$W = \int_0^2 3x\,dx$$

Step 2: Integrate.

$$= \left[\frac{3x^2}{2}\right]_0^2$$

Step 3: Evaluate at the bounds.

$$= \frac{3(4)}{2} - 0 = \frac{12}{2} = 6$$

$$\boxed{6}$$

18. Pumping water requires calculating work to overcome:

Every horizontal layer of water must be lifted to the top of the container. Deeper layers require more work because they have to travel farther against gravity.

$$\boxed{\text{Gravity and the weight of water at each depth}}$$

19. The average value of $f(x)$ on $[a, b]$ is given by:

Divide the total accumulated value of f (the integral) by the width of the interval.

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

20. Find the average value of $f(x) = x^2$ on $[0, 3]$.

Step 1: Set up the average value formula. $a = 0$, $b = 3$, so $b - a = 3$.

$$f_{\text{avg}} = \frac{1}{3}\int_0^3 x^2\,dx$$

Step 2: Integrate.

$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3$$

Step 3: Evaluate at the bounds.

$$= \frac{1}{3}\left(\frac{27}{3} - 0\right) = \frac{1}{3}(9) = 3$$

$$\boxed{3}$$

21. Find the average value of $f(x) = \sin(x)$ on $[0, \pi]$.

Step 1: Set up the average value formula. $b - a = \pi - 0 = \pi$.

$$f_{\text{avg}} = \frac{1}{\pi}\int_0^\pi \sin(x)\,dx$$

Step 2: Integrate. Recall $\int \sin(x)\,dx = -\cos(x) + C$**.**

$$= \frac{1}{\pi}\Big[-\cos(x)\Big]_0^\pi$$

Step 3: Evaluate at the bounds.

$$= \frac{1}{\pi}\big(-\cos(\pi) + \cos(0)\big) = \frac{1}{\pi}\big(-(-1) + 1\big) = \frac{1}{\pi}(2) = \frac{2}{\pi}$$

$$\boxed{\frac{2}{\pi}}$$

22. Find the area between $y = e^x$ and $y = 1$ from $x = -1$ to $x = 0$.

Step 1: Determine which curve is on top. On $[-1, 0]$, e^x is always positive but less than or equal to 1 (since $e^0 = 1$ and e^x is increasing). Check at $x = -0.5$: $e^{-0.5} \approx 0.607 < 1$. So $y = 1$ is on top.

Step 2: Set up the integral.

$$A = \int_{-1}^{0} \left(1 - e^x\right) dx$$

Step 3: Integrate term by term.

$$= \Big[x - e^x\Big]_{-1}^{0}$$

Step 4: Evaluate at the upper bound $x = 0$.

$$(0 - e^0) = (0 - 1) = -1$$

Step 5: Evaluate at the lower bound $x = -1$.

$$(-1 - e^{-1}) = -1 - \frac{1}{e}$$

Step 6: Subtract lower from upper.

$$A = (-1) - \left(-1 - \frac{1}{e}\right) = -1 + 1 + \frac{1}{e} = \frac{1}{e}$$

$$\boxed{\frac{1}{e}}$$

23. Find the volume when $y = \dfrac{1}{x}$ is revolved around the x-axis from $x = 1$ to $x = 2$.

Step 1: Use the disk method. The radius is $f(x) = \dfrac{1}{x}$.

$$V = \pi \int_{1}^{2} \left(\frac{1}{x}\right)^2 dx = \pi \int_{1}^{2} x^{-2}\, dx$$

Step 2: Integrate.

$$= \pi \left[-x^{-1}\right]_1^2 = \pi \left[-\frac{1}{x}\right]_1^2$$

Step 3: Evaluate at the bounds.

$$= \pi \left(-\frac{1}{2} - (-1)\right) = \pi \left(-\frac{1}{2} + 1\right) = \pi \cdot \frac{1}{2} = \frac{\pi}{2}$$

$$\boxed{\frac{\pi}{2}}$$

24. A solid has a triangular cross-section perpendicular to the x-axis with base $b(x)$ and height $h(x)$. The volume is:

The area of a triangle is $\frac{1}{2} \cdot \text{base} \cdot \text{height}$. Integrating these triangular cross-sections along x gives the total volume.

$$\boxed{V = \int_a^b \frac{b(x) \cdot h(x)}{2}\, dx}$$

25. Find the area under $y = e^{-x}$ from $x = 0$ to $x = \infty$.

Step 1: Recognize this as an improper integral. The upper limit is ∞, so we replace it with a variable t and take a limit.

$$\int_0^\infty e^{-x}\, dx = \lim_{t\to\infty} \int_0^t e^{-x}\, dx$$

Step 2: Integrate. Recall $\int e^{-x}\, dx = -e^{-x} + C$.

$$= \lim_{t\to\infty} \left[-e^{-x}\right]_0^t$$

Step 3: Evaluate at the bounds.

$$= \lim_{t\to\infty} \left(-e^{-t} + e^0\right) = \lim_{t\to\infty} \left(-e^{-t} + 1\right)$$

Step 4: Take the limit. As $t \to \infty$, $e^{-t} \to 0$.

$$= -0 + 1 = 1$$

$$\boxed{1}$$

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