Calculus Integrals Quiz
Answer Key
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1. A Riemann sum approximates the area under a curve by:

Explanation: To approximate the area between a curve $y = f(x)$ and the x-axis over $[a, b]$, we divide the interval into n sub-intervals of width $\Delta x = \dfrac{b-a}{n}$. On each sub-interval, we build a rectangle whose height is the function value at a chosen sample point. The Riemann sum is the total area of all these rectangles:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\boxed{\text{Dividing the region into rectangles and summing their areas}}$$

2. The limit of Riemann sums as $\Delta x \to 0$ defines:

Explanation: As the number of rectangles $n \to \infty$ (and their widths $\Delta x \to 0$), the approximation becomes exact. This limiting process defines the definite integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\boxed{\text{The definite integral}}$$

3. The Fundamental Theorem of Calculus Part 1 states:

Explanation: Part 1 of the Fundamental Theorem establishes that differentiation and integration are inverse operations. If $F(x) = \displaystyle\int_a^x f(t)\,dt$, then:

$$F'(x) = f(x)$$

Differentiating an integral of f recovers f itself. This connects the two central concepts of calculus.

$$\boxed{\text{The derivative of an integral recovers the original function}}$$

4. Evaluate $\displaystyle\int_1^3 2x\,dx$.

Step 1: Find the antiderivative. The antiderivative of $2x$ is x^2.

Step 2: Apply the Fundamental Theorem (Part 2): evaluate at the upper limit minus the lower limit.

$$\int_1^3 2x\,dx = \left[x^2\right]_1^3 = 3^2 - 1^2 = 9 - 1 = 8$$

$$\boxed{8}$$

5. Evaluate $\int 5x^4\,dx$.

Step 1: Apply the Power Rule for integration: increase the exponent by 1, then divide by the new exponent.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

Step 2: Apply with $n = 4$.

$$\int 5x^4\,dx = 5 \cdot \frac{x^5}{5} + C = x^5 + C$$

$$\boxed{x^5 + C}$$

6. Evaluate $\int \frac{1}{x}\,dx$.

Explanation: The Power Rule breaks down for $n = -1$ (it would require dividing by zero). Instead, the integral of $\frac{1}{x}$ is the natural logarithm. Absolute value is used to handle both positive and negative x:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\boxed{\ln|x| + C}$$

7. Which is a property of definite integrals?

Explanation: Definite integrals satisfy several key algebraic properties:

- **Linearity (scalar):** $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- **Linearity (sum):** $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- **Interval splitting:** $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

All of these are valid properties.

$$\boxed{\text{All of the above}}$$

8. If $\int_0^2 f(x)\,dx = 4$ and $\int_0^2 g(x)\,dx = 2$, find $\int_0^2 [f(x) + 3g(x)]\,dx$.
Step 1: Apply the linearity property of integrals — split and factor constants.

$$\int_0^2 [f(x) + 3g(x)]\,dx = \int_0^2 f(x)\,dx + 3\int_0^2 g(x)\,dx$$

Step 2: Substitute the given values.

$$= 4 + 3(2) = 4 + 6 = 10$$

$$\boxed{10}$$

9. Evaluate $\int (2x+1)^5 \cdot 2\,dx$.

Step 1: Use substitution. Let $u = 2x + 1$, so $du = 2\,dx$.
Step 2: Rewrite the integral entirely in terms of u.

$$\int u^5\,du$$

Step 3: Integrate using the Power Rule.

$$= \frac{u^6}{6} + C$$

Step 4: Substitute back $u = 2x + 1$.

$$= \frac{(2x+1)^6}{6} + C$$

$$\boxed{\frac{(2x+1)^6}{6} + C}$$

10. Evaluate $\int \sin(3x)\,dx$.

Step 1: Use substitution. Let $u = 3x$, so $du = 3\,dx$, meaning $dx = \dfrac{du}{3}$.
Step 2: Rewrite and integrate.

$$\int \sin(u) \cdot \frac{du}{3} = \frac{1}{3}\int \sin(u)\,du = \frac{1}{3}(-\cos u) + C$$

Step 3: Substitute back.

$$= -\frac{\cos(3x)}{3} + C$$

$$\boxed{-\frac{\cos(3x)}{3} + C}$$

11. Evaluate $\int 2x \cdot e^{x^2}\, dx$.

Step 1: Use substitution. Let $u = x^2$, so $du = 2x\, dx$.
Step 2: The factor $2x\, dx$ is exactly du, so the integral simplifies completely.

$$\int e^u\, du = e^u + C$$

Step 3: Substitute back $u = x^2$.

$$= e^{x^2} + C$$

$$\boxed{e^{x^2} + C}$$

12. Evaluate $\int_0^1 3x^2\sqrt{x^3+1}\, dx$.

Step 1: Use substitution. Let $u = x^3 + 1$, so $du = 3x^2\, dx$.
Step 2: Change the limits. When $x = 0$: $u = 1$. When $x = 1$: $u = 2$.
Step 3: Rewrite the integral.

$$\int_1^2 \sqrt{u}\, du = \int_1^2 u^{1/2}\, du$$

Step 4: Integrate and evaluate.

$$= \left[\frac{u^{3/2}}{\frac{3}{2}}\right]_1^2 = \left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{2}{3}(2)^{3/2} - \frac{2}{3}(1)^{3/2} = \frac{2}{3}(2\sqrt{2}) - \frac{2}{3} = \frac{4\sqrt{2}-2}{3} = \frac{2(2\sqrt{2}-1)}{3}$$

$$\boxed{\frac{2(2\sqrt{2}-1)}{3} \approx 1.219}$$

13. Evaluate $\int \sin^2(x)\, dx$.

Step 1: Use the half-angle identity to reduce the power:

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

Step 2: Rewrite and integrate term by term.

$$\int \frac{1-\cos(2x)}{2}\, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\, dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

$$\boxed{\frac{x}{2} - \frac{\sin(2x)}{4} + C}$$

14. Evaluate $\int \sin(x)\cos(x)\,dx$.

Step 1: Use substitution. Let $u = \sin(x)$, so $du = \cos(x)\,dx$.
Step 2: Rewrite and integrate.

$$\int u\,du = \frac{u^2}{2} + C$$

Step 3: Substitute back.

$$= \frac{\sin^2(x)}{2} + C$$

$$\boxed{\frac{\sin^2(x)}{2} + C}$$

15. Evaluate $\int \cos^2(x)\,dx$.

Step 1: Use the half-angle identity:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

Step 2: Integrate term by term.

$$\int \frac{1+\cos(2x)}{2}\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$$

$$\boxed{\frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

16. Evaluate $\int x\sin(x)\,dx$.

Step 1: This is a product of two different types of functions, so use Integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

Step 2: Choose $u = x$ and $dv = \sin(x)\,dx$.

$$du = dx \qquad v = -\cos(x)$$

Step 3: Apply the formula.

$$\int x\sin(x)\,dx = x(-\cos x) - \int(-\cos x)\,dx = -x\cos(x) + \int \cos(x)\,dx$$

Step 4: Integrate the remaining term.

$$= -x\cos(x) + \sin(x) + C$$

$$\boxed{-x\cos(x)+\sin(x)+C}$$

17. Evaluate $\displaystyle\int xe^x\,dx$.

Step 1: Use Integration by Parts with $u = x$ and $dv = e^x\,dx$.

$$du = dx \qquad v = e^x$$

Step 2: Apply the formula.

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Step 3: Factor.

$$= e^x(x-1)+C$$

$$\boxed{e^x(x-1)+C}$$

18. Evaluate $\displaystyle\int \sqrt{1-x^2}\,dx$.

Step 1: Use the trigonometric substitution $x = \sin\theta$, so $dx = \cos\theta\,d\theta$.
Step 2: Simplify the radical.

$$\sqrt{1-\sin^2\theta} = \cos\theta$$

Step 3: The integral becomes $\displaystyle\int \cos^2\theta\,d\theta$. Use the half-angle identity.

$$\int \frac{1+\cos(2\theta)}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$$

Step 4: Convert back using $\theta = \arcsin(x)$ and $\sin(2\theta) = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$.

$$= \frac{\arcsin(x)}{2} + \frac{x\sqrt{1-x^2}}{2} + C = \frac{1}{2}\Big[x\sqrt{1-x^2} + \arcsin(x)\Big] + C$$

$$\boxed{\frac{1}{2}\Big[x\sqrt{1-x^2} + \arcsin(x)\Big] + C}$$

19. Evaluate $\int \frac{1}{x^2-1}\,dx$.

Step 1: Use partial fractions. Factor the denominator: $x^2 - 1 = (x-1)(x+1)$.

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Step 2: Solve for A and B by multiplying through by $(x-1)(x+1)$.

$$1 = A(x+1) + B(x-1)$$

Set $x = 1$: $1 = 2A \Rightarrow A = \frac{1}{2}$. Set $x = -1$: $1 = -2B \Rightarrow B = -\frac{1}{2}$.

Step 3: Rewrite and integrate.

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

Step 4: Combine using log properties.

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

$$\boxed{\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$$

20. Evaluate $\int \frac{2x+1}{x^2+x-2}\,dx$.

Step 1: Recognize that the numerator $2x+1$ is the derivative of the denominator $x^2 + x - 2$. Confirm:

$$\frac{d}{dx}[x^2 + x - 2] = 2x + 1 \quad \checkmark$$

Step 2: This matches the form $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

$$\int \frac{2x+1}{x^2+x-2}\,dx = \ln|x^2 + x - 2| + C$$

Step 3: Factor the denominator to write in the equivalent form: $x^2+x-2 = (x+2)(x-1)$.

$$= \ln|(x+2)(x-1)| + C$$

$$\boxed{\ln|(x+2)(x-1)| + C}$$

21. Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$.

Step 1: Replace the infinite upper limit with a parameter b and take the limit.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b x^{-2}\,dx$$

Step 2: Find the antiderivative: $\int x^{-2}\,dx = -x^{-1} = -\frac{1}{x}$.
Step 3: Evaluate and take the limit.

$$= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 0+1 = 1$$

$$\boxed{1}$$

22. Evaluate $\displaystyle\int_0^\infty e^{-x}\,dx$.

Step 1: Replace the infinite limit with b and take the limit.

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$

Step 2: Integrate: $\int e^{-x}\,dx = -e^{-x}$.
Step 3: Evaluate and take the limit.

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left(-e^{-b}+e^0\right) = 0+1 = 1$$

$$\boxed{1}$$

23. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

Step 1: Split at 0 and replace both infinite limits with parameters.

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$$

Step 2: Recall that $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$.
Step 3: Evaluate each piece.

$$= [\arctan(0) - \arctan(-\infty)] + [\arctan(\infty) - \arctan(0)]$$

$$= \left[0 - \left(-\frac{\pi}{2}\right)\right] + \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\boxed{\pi}$$

24. Evaluate $\int \sec^2(x)\,dx$.

Explanation: This is a standard antiderivative. Since $\frac{d}{dx}[\tan(x)] = \sec^2(x)$, the integral is simply:

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\boxed{\tan(x) + C}$$

25. Evaluate $\int_0^{\pi/4} \sec(x)\tan(x)\,dx$.

Step 1: Recall the standard antiderivative: $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$, so:

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

Step 2: Evaluate at the limits.

$$\int_0^{\pi/4} \sec(x)\tan(x)\,dx = [\sec(x)]_0^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec(0)$$

Step 3: Use exact values: $\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos(\pi/4)} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$ and $\sec(0) = 1$.

$$= \sqrt{2} - 1$$

$$\boxed{\sqrt{2} - 1 \approx 0.414}$$

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