Calculus Derivatives Quiz
Answer Key
ApexMath

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1. The derivative of $f(x)$ at $x = a$ using the limit definition is:

Explanation: The derivative is defined as the limit of the difference quotient. As $h \to 0$, the average rate of change over the interval $[a,\ a+h]$ becomes the instantaneous rate of change at a:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$$

2. The derivative represents:

Explanation: The derivative $f'(x)$ simultaneously represents several equivalent ideas:

- The **instantaneous rate of change** of f at x
- The **slope of the tangent line** to the graph of f at x
- The **limit of the average rate of change** as the interval shrinks to zero

All of these descriptions are correct and equivalent.

$$\boxed{\text{All of the above}}$$

3. If $f'(x)$ exists at $x = a$, then f must be:

Explanation: Differentiability at $x = a$ implies:

- **Continuous at** $x = a$: a function cannot be differentiable at a point of discontinuity.
- **Defined at** $x = a$: the derivative cannot exist where the function itself is undefined.

Note that continuity does not guarantee differentiability (e.g. a corner), but differentiability does guarantee both of the above.

$$\boxed{\text{Continuous at } x = a \text{ and defined at } x = a}$$

4. The power rule states that if $f(x) = x^n$, then $f'(x)$ equals:

Explanation: The Power Rule is the most frequently used differentiation rule. Bring the exponent down as a coefficient, then reduce the exponent by 1:

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

This works for any real number n, including negative and fractional exponents.

$$\boxed{f'(x) = nx^{n-1}}$$

5. The derivative of $f(x) = 7$ (a constant) is:

Explanation: A constant function has a horizontal graph — it never rises or falls. Its slope is always 0, so its derivative is 0. This also follows from the Power Rule: $7 = 7x^0$, and $\dfrac{d}{dx}[7x^0] = 7 \cdot 0 \cdot x^{-1} = 0$.

$$\boxed{f'(x) = 0}$$

6. The sum/difference rule states that $[f(x) + g(x)]'$ equals:

Explanation: Differentiation is a linear operation. Derivatives can be applied term by term to sums and differences:

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

$$\boxed{f'(x) + g'(x)}$$

7. Find the derivative of $f(x) = 4x^3 - 2x^2 + 5x - 1$.

Step 1: Apply the Power Rule to each term separately.

$$\frac{d}{dx}[4x^3] = 12x^2 \qquad \frac{d}{dx}[-2x^2] = -4x \qquad \frac{d}{dx}[5x] = 5 \qquad \frac{d}{dx}[-1] = 0$$

Step 2: Combine all terms.

$$f'(x) = 12x^2 - 4x + 5$$

$$\boxed{f'(x) = 12x^2 - 4x + 5}$$

8. The derivative of $f(x) = \sqrt{x}$ is:

Step 1: Rewrite using a rational exponent: $\sqrt{x} = x^{1/2}$.
Step 2: Apply the Power Rule.

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

9. The product rule states that $[f(x)g(x)]'$ equals:

Explanation: When two functions are multiplied together, you cannot simply multiply their derivatives. The Product Rule is:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

A helpful mnemonic: "derivative of first times second, plus first times derivative of second."

$$\boxed{f'(x)g(x) + f(x)g'(x)}$$

10. Find the derivative of $f(x) = (x^2 + 1)(x^3 - 2)$.

Step 1: Identify $f(x) = x^2 + 1$ and $g(x) = x^3 - 2$. Find each derivative.

$$f'(x) = 2x \qquad g'(x) = 3x^2$$

Step 2: Apply the Product Rule.

$$[f \cdot g]' = (2x)(x^3 - 2) + (x^2 + 1)(3x^2)$$

Step 3: Expand each product.

$$= 2x^4 - 4x + 3x^4 + 3x^2$$

Step 4: Combine like terms.

$$= 5x^4 + 3x^2 - 4x$$

$$\boxed{f'(x) = 5x^4 + 3x^2 - 4x}$$

11. The quotient rule states that $\left[\dfrac{f(x)}{g(x)}\right]'$ equals:

Explanation: The Quotient Rule handles derivatives of fractions. The numerator follows a "hi-d-lo minus lo-d-hi" pattern, all divided by the square of the denominator:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Note the order of subtraction in the numerator: it is not commutative.

$$\boxed{\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

12. Find the derivative of $f(x) = \dfrac{2x+1}{x-3}$.

Step 1: Identify the numerator $u = 2x + 1$ and denominator $v = x - 3$.

$$u' = 2 \qquad v' = 1$$

Step 2: Apply the Quotient Rule.

$$f'(x) = \frac{u'v - uv'}{v^2} = \frac{(2)(x-3) - (2x+1)(1)}{(x-3)^2}$$

Step 3: Expand the numerator.

$$= \frac{2x - 6 - 2x - 1}{(x-3)^2} = \frac{-7}{(x-3)^2}$$

$$\boxed{f'(x) = \frac{-7}{(x-3)^2}}$$

13. The chain rule states that $[f(g(x))]'$ equals:

Explanation: The Chain Rule handles composite functions (functions inside other functions). Differentiate the outer function, leave the inner function unchanged, then multiply by the derivative of the inner function:

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

$$\boxed{f'(g(x)) \cdot g'(x)}$$

14. Find the derivative of $f(x) = (3x^2 + 2)^5$.

Step 1: Identify the outer function $f(u) = u^5$ and inner function $g(x) = 3x^2 + 2$.

$$f'(u) = 5u^4 \qquad g'(x) = 6x$$

Step 2: Apply the Chain Rule: outer derivative times inner derivative.

$$f'(x) = 5(3x^2 + 2)^4 \cdot 6x = 30x(3x^2 + 2)^4$$

$$\boxed{f'(x) = 30x(3x^2 + 2)^4}$$

15. Find the derivative of $f(x) = \sqrt{5x + 1}$.

Step 1: Rewrite as $f(x) = (5x+1)^{1/2}$. The outer function is $u^{1/2}$ and the inner function is $5x + 1$.

Step 2: Apply the Chain Rule.

$$f'(x) = \frac{1}{2}(5x + 1)^{-1/2} \cdot 5 = \frac{5}{2\sqrt{5x + 1}}$$

$$\boxed{f'(x) = \frac{5}{2\sqrt{5x + 1}}}$$

16. The derivative of $f(x) = e^x$ is:

Explanation: The natural exponential function e^x is unique — it is its own derivative. This is the defining property of $e \approx 2.71828$ and makes exponential functions fundamental in differential equations.

$$\frac{d}{dx}[e^x] = e^x$$

$$\boxed{f'(x) = e^x}$$

17. The derivative of $f(x) = \ln(x)$ is:

Explanation: The natural logarithm and the natural exponential are inverse functions, and their derivatives reflect this relationship. For $x > 0$:

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

$$\boxed{f'(x) = \frac{1}{x}}$$

18. Find the derivative of $f(x) = e^{3x}$.

Step 1: This is a composite function with outer function e^u and inner function $u = 3x$.

Step 2: Apply the Chain Rule. The derivative of e^u is e^u, and the derivative of $3x$ is 3.

$$f'(x) = e^{3x} \cdot 3 = 3e^{3x}$$

$$\boxed{f'(x) = 3e^{3x}}$$

19. Find the derivative of $f(x) = \ln(2x)$.

Step 1: Apply the Chain Rule with outer function $\ln(u)$ and inner function $u = 2x$.

$$f'(x) = \frac{1}{2x} \cdot 2 = \frac{2}{2x} = \frac{1}{x}$$

Note: Alternatively, use the log property: $\ln(2x) = \ln(2) + \ln(x)$. Since $\ln(2)$ is a constant, its derivative is 0, and $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$.

$$\boxed{f'(x) = \frac{1}{x}}$$

20. The derivative of $f(x) = a^x$ (where $a > 0,\ a \neq 1$) is:

Explanation: For a general exponential base a, the derivative introduces a factor of $\ln(a)$:

$$\frac{d}{dx}[a^x] = a^x \ln(a)$$

When $a = e$, we get $\ln(e) = 1$, so the formula reduces to $\frac{d}{dx}[e^x] = e^x$, consistent with Q16.

$$\boxed{f'(x) = a^x \ln(a)}$$

21. The derivative of $f(x) = \sin(x)$ is:

Explanation: This is a standard result derived from the limit definition using the special limit $\lim_{h \to 0} \frac{\sin h}{h} = 1$.

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\boxed{f'(x) = \cos(x)}$$

22. The derivative of $f(x) = \cos(x)$ is:

Explanation: The cosine derivative introduces a negative sign:

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

One way to remember the pattern: differentiating the co-functions (cosine, cotangent, cosecant) always produces a negative sign.

$$\boxed{f'(x) = -\sin(x)}$$

23. The derivative of $f(x) = \tan(x)$ is:

Step 1: Derive using the Quotient Rule: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

$$\frac{d}{dx}[\tan(x)] = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

Step 2: Apply the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$.

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$\boxed{f'(x) = \sec^2(x)}$$

24. The derivative of $f(x) = \cot(x)$ is:

Step 1: Write $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and apply the Quotient Rule.

$$\frac{d}{dx}[\cot(x)] = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} = \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)}$$

Step 2: Since $\frac{1}{\sin(x)} = \csc(x)$:

$$= -\csc^2(x)$$

$$\boxed{f'(x) = -\csc^2(x)}$$

25. The derivative of $f(x) = \sec(x)$ is:

Step 1: Write $\sec(x) = \dfrac{1}{\cos(x)}$ and apply the Quotient Rule.

$$\frac{d}{dx}[\sec(x)] = \frac{0 \cdot \cos(x) - 1 \cdot (-\sin(x))}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)}$$

Step 2: Factor to match the standard form.

$$= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$

$$\boxed{f'(x) = \sec(x)\tan(x)}$$

26. The derivative of $f(x) = \csc(x)$ is:

Step 1: Write $\csc(x) = \dfrac{1}{\sin(x)}$ and apply the Quotient Rule.

$$\frac{d}{dx}[\csc(x)] = \frac{0 \cdot \sin(x) - 1 \cdot \cos(x)}{\sin^2(x)} = \frac{-\cos(x)}{\sin^2(x)}$$

Step 2: Factor into standard form.

$$= -\frac{1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$$

$$\boxed{f'(x) = -\csc(x)\cot(x)}$$

27. The derivative of $f(x) = \arcsin(x)$ is:

Explanation: Using implicit differentiation on $y = \arcsin(x)$ (i.e. $\sin y = x$):

$$\cos(y)\frac{dy}{dx} = 1 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Since $\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$ (for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$):

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{f'(x) = \frac{1}{\sqrt{1 - x^2}}}$$

28. The derivative of $f(x) = \arctan(x)$ is:

Explanation: Using implicit differentiation on $y = \arctan(x)$ (i.e. $\tan y = x$):

$$\sec^2(y)\frac{dy}{dx} = 1 \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y)$$

Since $\sec^2(y) = 1 + \tan^2(y) = 1 + x^2$:

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$$

$$\boxed{f'(x) = \frac{1}{1+x^2}}$$

29. The second derivative $f''(x)$ is defined as:

Explanation: The second derivative is obtained by differentiating $f'(x)$ once more. It measures the rate of change of the rate of change — in other words, how quickly the slope is changing. It can be written equivalently as:

$$f''(x) = \frac{d}{dx}[f'(x)] = \frac{d^2 f}{dx^2} = (f')'(x)$$

All of these notations describe the same object.

$$\boxed{\text{All of the above (equivalent notations for the second derivative)}}$$

30. If $f(x) = x^4 - 3x^3 + 2x$, find $f''(x)$.

Step 1: Find the first derivative using the Power Rule.

$$f'(x) = 4x^3 - 9x^2 + 2$$

Step 2: Differentiate again to find the second derivative.

$$f''(x) = 12x^2 - 18x$$

$$\boxed{f''(x) = 12x^2 - 18x}$$

31. When using implicit differentiation on an equation like $x^2 + y^2 = 25$:

Explanation: In implicit differentiation, y is treated as an unknown function of x (i.e. $y = y(x)$). Both sides of the equation are differentiated with respect to x. When differentiating a term involving y, the Chain Rule applies: every derivative of y picks up a factor of $\frac{dy}{dx}$.

$$\boxed{\text{Differentiate both sides with respect to } x, \text{ treating } y \text{ as a function of } x}$$

32. Using implicit differentiation on $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Step 1: Differentiate both sides with respect to x. Apply the Chain Rule to the y^2 term.

$$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[25]$$

$$2x + 2y\frac{dy}{dx} = 0$$

Step 2: Isolate $\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = -2x \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{x}{y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

33. In a related rates problem, we use the chain rule to find:

Explanation: Related rates problems involve quantities that both depend on time t. By differentiating both sides of an equation with respect to t (using the Chain Rule), we find how the rates of change of the quantities are related to each other.

$$\boxed{\text{How quantities related by an equation change over time}}$$

34. A 10-foot ladder leans against a wall. The bottom slides away at 2 ft/s. How fast is the top sliding down when the bottom is 6 feet from the wall?

Step 1: Let x = horizontal distance (bottom from wall) and y = vertical height (top on wall). The constraint is $x^2 + y^2 = 100$.
Step 2: Find y when $x = 6$.

$$6^2 + y^2 = 100 \qquad \Rightarrow \qquad y^2 = 64 \qquad \Rightarrow \qquad y = 8 \text{ ft}$$

Step 3: Differentiate $x^2 + y^2 = 100$ with respect to time t.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Step 4: Substitute $x = 6$, $y = 8$, and $\dfrac{dx}{dt} = 2$ ft/s.

$$2(6)(2) + 2(8)\frac{dy}{dt} = 0 \qquad \Rightarrow \qquad 24 + 16\frac{dy}{dt} = 0 \qquad \Rightarrow \qquad \frac{dy}{dt} = -\frac{24}{16} = -1.5 \text{ ft/s}$$

The negative sign indicates the top is moving downward. The speed is 1.5 ft/s.

$$\boxed{\frac{dy}{dt} = 1.5 \text{ ft/s downward}}$$

35. A critical point occurs where:

Explanation: Critical points are candidates for local maxima and minima. A critical point occurs at $x = c$ if either:

- $f'(c) = 0$ (the tangent line is horizontal), or
- $f'(c)$ is undefined (the function has a corner, cusp, or vertical tangent)

Not every critical point is a maximum or minimum — it is simply a place where the derivative fails to be strictly positive or strictly negative.

$$\boxed{f'(x) = 0 \text{ or } f'(x) \text{ is undefined}}$$

36. If $f'(x)$ changes from positive to negative at $x = c$, then f has a:

Explanation: This is the **First Derivative Test**. When $f'(x)$ goes from positive (increasing) to negative (decreasing) as x passes through c, the function has a peak at $x = c$ — a local maximum.

- $f' > 0$ then $f' < 0$: local **maximum**
- $f' < 0$ then $f' > 0$: local **minimum**
- No sign change: neither (saddle point)

$$\boxed{\text{Local maximum at } x = c}$$

37. A function is concave up on an interval when:

Explanation: Concavity describes the curvature of the graph. When $f''(x) > 0$, the slope $f'(x)$ is increasing — the graph curves upward like a bowl (or a smile). When $f''(x) < 0$, the graph curves downward like a hill (or a frown).

$$\boxed{f''(x) > 0}$$

38. An inflection point occurs where:

Explanation: An inflection point is where the concavity of a function changes (from concave up to concave down, or vice versa). This requires $f''(x) = 0$ or $f''(x)$ undefined, but these are only candidates — the concavity must actually change to confirm an inflection point.

$$\boxed{f''(x) = 0 \text{ or } f''(x) \text{ is undefined}}$$

39. To find the maximum or minimum of a function on a closed interval $[a,\ b]$:

Explanation: On a closed interval, the absolute (global) maximum and minimum can occur at interior critical points or at the endpoints. The procedure is:

1. Find all x in (a, b) where $f'(x) = 0$ or $f'(x)$ is undefined.
2. Evaluate f at each critical point and at both endpoints a and b.
3. The largest value is the absolute maximum; the smallest is the absolute minimum.

$$\boxed{\text{Check critical points and endpoints}}$$

40. A box with open top, square base $x \times x$, and volume 500 in^3 has surface area $S(x) = x^2 + \dfrac{2000}{x}$. Find the critical point that minimizes S.

Step 1: Differentiate $S(x)$.

$$S'(x) = 2x - \frac{2000}{x^2}$$

Step 2: Set $S'(x) = 0$ and solve.

$$2x = \frac{2000}{x^2} \qquad \Rightarrow \qquad 2x^3 = 2000 \qquad \Rightarrow \qquad x^3 = 1000 \qquad \Rightarrow \qquad x = 10$$

Step 3: Confirm it is a minimum using the second derivative.

$$S''(x) = 2 + \frac{4000}{x^3} \qquad \Rightarrow \qquad S''(10) = 2 + 4 = 6 > 0 \quad \checkmark$$

Since $S''(10) > 0$, the function is concave up at $x = 10$, confirming a minimum.

$$\boxed{x = 10 \text{ inches}}$$

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