Differential Equations — Answer Key
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1. A differential equation is an equation that contains **a function and its derivatives**.

Reason: By definition, a differential equation relates an unknown function to one or more of its derivatives.

2. The order of a differential equation is determined by **the highest derivative present**.

Reason: For example, an equation containing $\dfrac{d^2y}{dx^2}$ but no higher derivative is called a *second-order* differential equation.

3. The equation that is first-order is $\dfrac{dy}{dx} = 2x + 1$.

Reason: It contains only the first derivative $\dfrac{dy}{dx}$, so its order is 1.

4. A slope field shows **small line segments whose slopes are determined by the differential equation**.

Reason: At each point (x, y), the slope $\dfrac{dy}{dx}$ is computed from the equation, and a short segment with that slope is drawn. Together, the segments visualise the family of solution curves.

5. At the point (2, 3) on the slope field for $\dfrac{dy}{dx} = x - y$, the slope is $\mathbf{-1}$.

Step 1: Identify the differential equation and the given point.

$$\frac{dy}{dx} = x - y, \qquad (x,\, y) = (2,\, 3)$$

Step 2: Substitute $x = 2$ and $y = 3$ into the right-hand side.

$$\frac{dy}{dx} = 2 - 3$$

Step 3: Simplify.

$$\boxed{-1}$$

6. Solutions that move *away* from an equilibrium describe an **unstable equilibrium**.

Reason: At an unstable equilibrium, small perturbations grow over time rather than returning to the equilibrium value.

7. To solve $\frac{dy}{dx} = 2xy$ by separation of variables, rearrange to $\mathbf{\frac{dy}{y} = 2x\,dx}$.

Step 1: Start with the equation.

$$\frac{dy}{dx} = 2xy$$

Step 2: Move all y terms to the left and all x terms to the right. Divide both sides by y and multiply both sides by dx.

$$\boxed{\frac{dy}{y} = 2x\,dx}$$

8. After separating variables in $\frac{dy}{dx} = 3y$, the next step is to **integrate both sides**.

Reason: Separating gives $\frac{dy}{y} = 3\,dx$. Both sides must then be integrated in order to remove the derivative and find the general solution.

9. Solving $\frac{1}{y}\,dy = x^2\,dx$ gives $\ln|\mathbf{y}| = \mathbf{\frac{x^3}{3} + C}$.

Step 1: Write down the separated equation.

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

Step 2: Integrate the left side. Recall $\int \frac{1}{y}\,dy = \ln|y|$.

$$\ln|y|$$

Step 3: Integrate the right side. Recall $\int x^2\,dx = \frac{x^3}{3}$.

$$\frac{x^3}{3} + C$$

Step 4: Combine both sides. The constant of integration C appears on one side only.

$$\boxed{\ln|y| = \frac{x^3}{3} + C}$$

10. A particular solution is determined by **the initial condition (or boundary condition)**.

Reason: The general solution contains an arbitrary constant C. An initial condition (a specific point (x_0, y_0) the solution must pass through) fixes the value of C, giving the unique particular solution.

11. Given $\dfrac{dy}{dx} = 4x$ and $y(1) = 5$, the particular solution is $\mathbf{y = 2x^2 + 3}$.

Step 1: Integrate both sides with respect to x.

$$\int dy = \int 4x\,dx$$

Step 2: Evaluate each integral.

$$y = 2x^2 + C$$

Step 3: Apply the initial condition $y(1) = 5$. Substitute $x = 1$ and $y = 5$.

$$5 = 2(1)^2 + C$$

Step 4: Solve for C.

$$5 = 2 + C \implies C = 3$$

Step 5: Write the particular solution.

$$\boxed{y = 2x^2 + 3}$$

12. For $y = e^x + C$ with $y(0) = 2$, we get $\mathbf{C = 1}$.

Step 1: Write down the general solution.

$$y = e^x + C$$

Step 2: Apply the initial condition $y(0) = 2$. Substitute $x = 0$ and $y = 2$.

$$2 = e^0 + C$$

Step 3: Recall that $e^0 = 1$.

$$2 = 1 + C$$

Step 4: Solve for C.

$$\boxed{C = 1}$$

13. The equation $\dfrac{dy}{dt} = ky$ describes **exponential growth or decay**.

Reason: Its solution is $y = Ce^{kt}$. When $k > 0$ the quantity grows without bound; when $k < 0$ it decays toward zero.

14. The general solution to $\dfrac{dy}{dt} = 2y$ is $\mathbf{y = Ce^{2t}}$.

Step 1: Separate variables. Divide both sides by y and multiply by dt.

$$\frac{dy}{y} = 2\,dt$$

Step 2: Integrate both sides.

$$\int \frac{1}{y}\,dy = \int 2\,dt$$

Step 3: Evaluate the integrals.

$$\ln|y| = 2t + C_1$$

Step 4: Exponentiate both sides to solve for y.

$$|y| = e^{2t+C_1} = e^{C_1} \cdot e^{2t}$$

Step 5: Replace e^{C_1} with the arbitrary constant C (which absorbs the $\pm$ from the absolute value).

$$\boxed{y = Ce^{2t}}$$

15. With $\dfrac{dy}{dt} = 0.05y$ and $y(0) = 1000$, the population at $t = 10$ is approximately **1649**.

Step 1: Identify the form of the solution. Since $\dfrac{dy}{dt} = ky$, the general solution is $y = Ce^{kt}$.

$$y = Ce^{0.05t}$$

Step 2: Apply the initial condition $y(0) = 1000$.

$$1000 = Ce^{0} = C \implies C = 1000$$

Step 3: Write the particular solution.

$$y = 1000\,e^{0.05t}$$

Step 4: Substitute $t = 10$.

$$y(10) = 1000\,e^{0.05\times 10} = 1000\,e^{0.5}$$

Step 5: Evaluate. Using $e^{0.5} \approx 1.6487$.

$$\boxed{y(10) \approx 1649}$$

16. Euler's method approximates solutions by **using tangent-line approximations at discrete points**.

Reason: At each step, the slope of the tangent line (given by $f(x_n, y_n)$) is used to project forward by a small step Δx, producing the next approximate value.

17. In the formula $y_{n+1} = y_n + f(x_n, y_n)\,\Delta x$, Δx represents **the step size**.

Reason: Δx is the horizontal distance between successive approximation points. A smaller Δx means more steps and a more accurate result.

18. Using Euler's method with $\dfrac{dy}{dx} = x + y$, $y(0) = 1$, and $\Delta x = 0.1$, the value $y_1 = \mathbf{1.1}$.

Step 1: Write down the Euler formula.

$$y_{n+1} = y_n + f(x_n,\, y_n)\,\Delta x$$

Step 2: Identify the starting values.

$$x_0 = 0, \quad y_0 = 1, \quad \Delta x = 0.1$$

Step 3: Compute $f(x_0, y_0) = x_0 + y_0$.

$$f(0,\, 1) = 0 + 1 = 1$$

Step 4: Apply the Euler step.

$$y_1 = y_0 + f(x_0,\, y_0)\,\Delta x = 1 + (1)(0.1)$$

Step 5: Simplify.

$$\boxed{y_1 = 1.1}$$

19. To improve accuracy in Euler's method, you should **decrease the step size**.

Reason: Each tangent-line approximation introduces error. Using a smaller Δx means shorter tangent-line segments, so the approximation stays closer to the true curve.

20. In $\dfrac{dy}{dt} = ky\Big(1 - \dfrac{y}{L}\Big)$, the parameter L represents **the carrying capacity**.

Reason: As $y \to L$ the factor $\Big(1 - \dfrac{y}{L}\Big)$ approaches 0, so growth slows to zero. L is therefore the maximum sustainable population.

21. In $\dfrac{dy}{dt} = ry\Big(1 - \dfrac{y}{K}\Big)$, growth is fastest when $\mathbf{y = K/2}$.

Reason: The growth rate $r\,y(1 - y/K)$ is a downward-opening parabola in y. Its maximum occurs at the midpoint of $[0,\, K]$, which is $y = K/2$.

22. As $t \to \infty$ in a logistic model, the population **approaches the carrying capacity** K.

Reason: Once y is close to K, the growth rate is nearly zero. The population levels off and remains near K indefinitely.

23. The scenario best modelled by a logistic equation is a **fish population in a lake with limited resources**.

Reason: Logistic growth applies when a natural limit (food, space, etc.) caps the population. Pure exponential growth assumes unlimited resources, which does not hold for fish in a finite lake.

24. Separating variables in $\dfrac{dy}{dx} = e^{-y}$ gives $\mathbf{e^y\,dy = dx}$.

Step 1: Write the equation.

$$\frac{dy}{dx} = e^{-y}$$

Step 2: Multiply both sides by dx.

$$dy = e^{-y}\,dx$$

Step 3: Divide both sides by e^{-y}, which is the same as multiplying by e^y.

$$\boxed{e^y\,dy = dx}$$

25. Slope fields and Euler's method both **provide approximate visualisations of solutions**.

Reason: Neither gives an exact closed-form solution. A slope field shows the direction of the solution curves graphically; Euler's method generates numerical approximations at discrete points. Both are tools for understanding behaviour when an exact solution is difficult or impossible to obtain.

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