Limits & Continuity Quiz
Answer Key
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1. The limit of a function $f(x)$ as x approaches a is L if:

Explanation: The formal definition of a limit captures the idea that we can make $f(x)$ as close to L as we like, simply by choosing x close enough to a. Crucially, the function does not need to be defined at $x = a$ — the limit only describes the behavior as x approaches a, not at a.

$$\boxed{\text{The function values get arbitrarily close to } L \text{ as } x \text{ gets arbitrarily close to } a}$$

2. Which of the following is a limit law?

Explanation: The limit laws are a complete set of rules for computing limits algebraically. They include:

- **Sum Law:** $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- **Product Law:** $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- **Quotient Law:** $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$

All of the listed options are valid limit laws.

$$\boxed{\text{All of the above}}$$

3. If $\lim_{x\to 2} f(x) = 5$ and $\lim_{x\to 2} g(x) = 3$, find $\lim_{x\to 2}[f(x) - g(x)]$.

Step 1: Apply the Difference Law for limits. When both individual limits exist, the limit of the difference equals the difference of the limits.

$$\lim_{x\to 2}[f(x) - g(x)] = \lim_{x\to 2} f(x) - \lim_{x\to 2} g(x)$$

Step 2: Substitute the given values.

$$= 5 - 3 = 2$$

$\boxed{2}$

4. The limit of a constant function $\lim_{x \to a} c$ equals:

Explanation: A constant function $f(x) = c$ outputs the same value c for every input x, no matter what x is. Therefore, as x approaches a, the function values are always c — the limit is simply c.

$$\lim_{x \to a} c = c$$

$$\boxed{c}$$

5. To evaluate $\lim_{x \to 2}(x^2 + 3x - 1)$, you can:

Explanation: Polynomial functions are continuous everywhere — they have no holes, jumps, or asymptotes. For any polynomial $p(x)$, the limit as $x \to a$ equals the value at a:

$$\lim_{x \to 2}(x^2 + 3x - 1) = (2)^2 + 3(2) - 1 = 4 + 6 - 1 = 9$$

This technique is called direct substitution and is always valid for polynomials.

$$\boxed{\text{Use direct substitution}}$$

6. When evaluating $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$, direct substitution gives $\frac{0}{0}$. This is called a(n):

Explanation: The form $\frac{0}{0}$ is called an **indeterminate form**. It does not mean the limit is 0, undefined, or 1 — it simply means direct substitution cannot determine the limit. A different technique (such as factoring) is required. Other indeterminate forms include $\frac{\infty}{\infty}$, $0 \cdot \infty$, and $\infty - \infty$.

$$\boxed{\text{Indeterminate form}}$$

7. Simplify and evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

Step 1: Direct substitution yields $\frac{0}{0}$, so factor the numerator as a difference of squares.

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3}$$

Step 2: Cancel the common factor $(x - 3)$. This is valid because in a limit, x approaches 3 but never equals 3, so $x - 3 \neq 0$.

$$= x + 3$$

Step 3: Now apply direct substitution.

$$\lim_{x \to 3}(x + 3) = 3 + 3 = 6$$

$$\boxed{6}$$

8. The limit $\lim_{x \to 0} \frac{\sin x}{x}$ equals:

Explanation: This is one of the most important special limits in calculus. Direct substitution gives $\frac{0}{0}$, so it cannot be evaluated by simple algebra. Using the Squeeze Theorem (or geometric reasoning), it can be shown that the ratio $\frac{\sin x}{x}$ approaches 1 as $x \to 0$:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This result is fundamental to proving that the derivative of $\sin x$ is $\cos x$.

$$\boxed{1}$$

9. The left-hand limit $\lim_{x \to a^-} f(x)$ refers to:

Explanation: The notation $x \to a^-$ means x approaches a from below (from the left side of a on the number line), taking on values slightly less than a. The left-hand limit describes the behavior of $f(x)$ on the left side only, without any consideration of what happens to the right of a.

$$\boxed{\text{The limit as } x \text{ approaches } a \text{ from the left}}$$

10. For a limit $\lim_{x \to a} f(x)$ to exist:

Explanation: A two-sided limit exists only when both the left-hand and right-hand limits exist and are equal. If the function approaches different values from the left and right, the two-sided limit does not exist:

$$\lim_{x \to a} f(x) \text{ exists} \quad \Longleftrightarrow \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

$$\boxed{\text{The left-hand limit must equal the right-hand limit}}$$

11. When evaluating $\lim_{x \to \infty} \frac{3x^2 + 2x}{5x^2 - 1}$, the dominant terms are:

Explanation: As $x \to \infty$, the highest-degree terms in the numerator and denominator grow much faster than lower-degree terms. These leading terms dominate the behavior of the fraction. Here the dominant terms are $3x^2$ (numerator) and $5x^2$ (denominator).

$$\boxed{3x^2 \text{ and } 5x^2}$$

12. Evaluate $\lim_{x \to \infty} \frac{3x^2 + 2x}{5x^2 - 1}$.

Step 1: Divide every term in the numerator and denominator by x^2, the highest power present.

$$\frac{3x^2 + 2x}{5x^2 - 1} = \frac{3 + \frac{2}{x}}{5 - \frac{1}{x^2}}$$

Step 2: As $x \to \infty$, the terms $\frac{2}{x}$ and $\frac{1}{x^2}$ both approach 0.

$$\lim_{x \to \infty} \frac{3 + \frac{2}{x}}{5 - \frac{1}{x^2}} = \frac{3 + 0}{5 - 0} = \frac{3}{5}$$

$$\boxed{\frac{3}{5}}$$

13. If $\lim_{x \to \infty} f(x) = L$, the line $y = L$ is called a(n):

Explanation: When the output of a function approaches a fixed value L as x grows without bound, the graph gets arbitrarily close to the horizontal line $y = L$ but may never reach it. This line is called a **horizontal asymptote**.

$$\boxed{\text{Horizontal asymptote}}$$

14. A function is continuous at $x = a$ if:

Explanation: Continuity at a point requires three conditions to be met simultaneously:

1. $f(a)$ is defined (the function exists at $x = a$).
2. $\lim_{x \to a} f(x)$ exists (the two-sided limit exists).
3. $\lim_{x \to a} f(x) = f(a)$ (the limit equals the function value).

The third condition actually implies the first two, so it is the concise way to state the requirement.

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

15. Which type of discontinuity can be "removed" by redefining the function at one point?

Explanation: A **removable discontinuity** occurs when $\lim_{x \to a} f(x)$ exists, but either $f(a)$ is undefined or $f(a)$ does not equal the limit. Because the limit exists, the gap is just a single missing or misplaced point. It can be "patched" by defining (or redefining) $f(a) = \lim_{x \to a} f(x)$.

$$\boxed{\text{Removable discontinuity}}$$

16. A jump discontinuity occurs when:

Explanation: A **jump discontinuity** occurs when the left-hand and right-hand limits both exist but are not equal. The graph "jumps" from one value to another at $x = a$, with no path connecting the two sides. This cannot be repaired by redefining the function at a single point.

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

$$\boxed{\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)}$$

17. The Intermediate Value Theorem states: if f is continuous on $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then:

Explanation: The IVT captures the intuitive idea that a continuous function cannot "skip over" values. If you draw a continuous curve from $(a, f(a))$ to $(b, f(b))$, it must cross every horizontal line $y = N$ that lies between those two heights at least once.

Formally: there exists at least one value $c \in (a, b)$ such that $f(c) = N$.

$$\boxed{\text{There exists at least one } c \in (a, b) \text{ such that } f(c) = N}$$

18. For $f(x) = x^3 - 4x$ on $[0, 2]$ with $f(0) = 0$, $f(2) = 0$, and $f(1) = -3$, the IVT guarantees:

Step 1: Note that f is a polynomial, so it is continuous everywhere, including on $[0, 2]$.
Step 2: On the sub-interval $[1, 2]$:

$$f(1) = -3 \qquad f(2) = 0$$

The value -1 lies between -3 and 0, i.e. $-3 < -1 < 0$.

Step 3: By the IVT applied to $[1, 2] \subset (0, 2)$, there must exist at least one $c \in (1, 2)$ such that $f(c) = -1$.

$$\boxed{f(x) = -1 \text{ has a solution in } (0,\ 2)}$$

19. The slope of the tangent line to a curve at a point P is given by:

Explanation: The tangent line at a point P on a curve touches the curve at exactly that point and represents the direction the curve is heading at that instant. The slope of this line equals the **instantaneous rate of change** of the function at P, which is defined as the limit of the average rate of change over smaller and smaller intervals containing P.

$$\boxed{\text{The instantaneous rate of change at point } P}$$

20. To find the slope of the tangent line at $x = 2$ for $f(x) = x^2$, you would evaluate:

Explanation: The slope of the tangent line is found by computing the derivative using the limit definition:

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \qquad \text{or equivalently} \qquad f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

Both expressions define the same instantaneous rate of change at $x = 2$, so both options B and C are correct.

$$\boxed{\text{Both limit definitions (difference quotient as } h \to 0 \text{ or as } x \to 2\text{)}}$$

21. The slope of the tangent line to $f(x) = x^2$ at $x = 2$ is:

Step 1: Apply the limit definition of the derivative.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$$

Step 2: Expand $(2+h)^2$.

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h}$$

Step 3: Factor h from the numerator and cancel.

$$= \lim_{h \to 0} \frac{h(4+h)}{h} = \lim_{h \to 0} (4+h) = 4$$

$$\boxed{f'(2) = 4}$$

22. The derivative of $f(x)$ at $x = a$ is defined as:

Explanation: The derivative can be written in several equivalent ways, all expressing the same limiting process:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Both forms define the instantaneous rate of change at $x = a$. Since all listed options are equivalent expressions for this limit, all are correct.

$$\boxed{\text{All of the above (equivalent limit definitions)}}$$

23. Using the definition of the derivative, find $f'(x)$ for $f(x) = 3x + 2$.

Step 1: Set up the difference quotient.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Substitute $f(x+h) = 3(x+h) + 2 = 3x + 3h + 2$.

$$f'(x) = \lim_{h \to 0} \frac{(3x + 3h + 2) - (3x + 2)}{h} = \lim_{h \to 0} \frac{3h}{h}$$

Step 3: Cancel h (valid since $h \neq 0$ in a limit) and evaluate.

$$f'(x) = \lim_{h \to 0} 3 = 3$$

This makes sense: $f(x) = 3x+2$ is a line with slope 3, so its derivative (slope) is constant at 3.

$$\boxed{f'(x) = 3}$$

24. The difference quotient $\dfrac{f(x+h) - f(x)}{h}$ represents:

Explanation: The difference quotient measures the slope of the straight line connecting two points on the graph of f: the point $(x, f(x))$ and the point $(x+h, f(x+h))$. This connecting line is called a **secant line**. As $h \to 0$, the secant line approaches the tangent line, and the difference quotient approaches the derivative.

$$\boxed{\text{The slope of the secant line between two points on the graph}}$$

25. If $f'(a)$ does not exist, it means:

Explanation: The derivative $f'(a)$ fails to exist in several geometric situations. The most common are:

- **Vertical tangent:** The tangent line is vertical, so its slope is undefined (the difference quotient grows without bound).
- **Corner or cusp:** The left-hand and right-hand limits of the difference quotient are not equal, so the two-sided limit (the derivative) does not exist.
- **Discontinuity:** If f is not continuous at $x = a$, it cannot be differentiable there.

$$\boxed{\text{The tangent is vertical, or } f \text{ has a corner/cusp at } x = a}$$

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