Comprehensive Unit Test
Final Exam Prep — Answer Key
ApexMath

Answer Key

1. Convert 300° to radians in simplest form.

Step 1: Multiply by the conversion factor $\frac{\pi}{180°}$.

$$300° \times \frac{\pi}{180°} = \frac{300\pi}{180}$$

Step 2: Simplify by dividing numerator and denominator by 60.

$$\frac{300\pi}{180} = \frac{5\pi}{3}$$

$$\boxed{\frac{5\pi}{3} \text{ radians}}$$

2. What is the coordinate of the point on the unit circle at angle $\frac{7\pi}{6}$?

Step 1: Find the reference angle. Since $\pi < \frac{7\pi}{6} < \frac{3\pi}{2}$, the angle is in Quadrant III.

$$\theta_R = \frac{7\pi}{6} - \pi = \frac{\pi}{6} \quad (30°)$$

Step 2: The unit circle coordinates at $\frac{\pi}{6}$ are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Step 3: In Quadrant III, both coordinates are negative.

$$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

$$\boxed{\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)}$$

3. In a 30-60-90 triangle with hypotenuse 10, find the length of the side opposite the 30° angle.

Step 1: In a 30-60-90 triangle, the sides are in the ratio $1 : \sqrt{3} : 2$. The side opposite 30° is the shortest leg.

$$\frac{\text{short leg}}{\text{hypotenuse}} = \frac{1}{2}$$

Step 2: Solve for the short leg.

$$\text{short leg} = \frac{1}{2} \times 10 = 5$$

$$\boxed{5 \text{ units}}$$

4. An angle of 225° is in which quadrant, and what is its reference angle?

Step 1: Since $180^\circ < 225^\circ < 270^\circ$, the angle is in **Quadrant III**.
Step 2: For Quadrant III, subtract 180° to find the reference angle.

$$\theta_R = 225^\circ - 180^\circ = 45^\circ$$

$$\boxed{\text{Quadrant III; reference angle} = 45^\circ}$$

5. What are the amplitude and period of $f(x) = 4\sin(2x)$?

Step 1: Identify $A = 4$ and $B = 2$ from the general form $f(x) = A\sin(Bx)$.
Step 2: The amplitude is $|A|$.

$$\text{Amplitude} = |4| = 4$$

Step 3: The period is $\frac{2\pi}{B}$.

$$\text{Period} = \frac{2\pi}{2} = \pi$$

$$\boxed{\text{Amplitude} = 4, \quad \text{Period} = \pi}$$

6. What are the vertical asymptotes of $f(x) = \tan(x)$ on $[0,\ \pi)$?

Step 1: Vertical asymptotes of $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ occur where $\cos(x) = 0$.

Step 2: On $[0, \pi)$, cosine equals zero at $x = \dfrac{\pi}{2}$ only (since $x = \dfrac{3\pi}{2}$ is outside the interval).

$$\boxed{x = \frac{\pi}{2}}$$

7. Evaluate $\arcsin\left(\dfrac{\sqrt{3}}{2}\right)$.

Step 1: arcsin asks: "what angle in $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$ has a sine of $\frac{\sqrt{3}}{2}$?"

Step 2: From the 30-60-90 triangle, $\sin(60^\circ) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, and $\frac{\pi}{3}$ is in the valid range.

$$\boxed{\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}}$$

8. Find $\sin(105^\circ) = \sin(60^\circ + 45^\circ)$.

Step 1: Apply the sine sum formula: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

$$\sin(105^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$$

Step 2: Substitute exact values.

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

Step 3: Combine.

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\boxed{\sin(105^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

9. If $\cos(\theta) = \frac{2}{3}$, find $\cos(2\theta)$.

Step 1: Use the double angle identity $\cos(2\theta) = 2\cos^2(\theta) - 1$.

Step 2: Substitute $\cos(\theta) = \frac{2}{3}$.

$$\cos(2\theta) = 2\left(\frac{2}{3}\right)^2 - 1 = 2\cdot\frac{4}{9} - 1 = \frac{8}{9} - \frac{9}{9} = -\frac{1}{9}$$

$$\boxed{\cos(2\theta) = -\frac{1}{9}}$$

10. Solve $\cos(x) = -\frac{\sqrt{2}}{2}$ for $x \in [0,\ 2\pi)$.

Step 1: Find the reference angle where $\cos\theta = \frac{\sqrt{2}}{2}$.

$$\theta_R = \frac{\pi}{4}$$

Step 2: Cosine is negative in Quadrant II and Quadrant III.

- QII: $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$
- QIII: $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$$\boxed{x = \frac{3\pi}{4} \quad \text{and} \quad x = \frac{5\pi}{4}}$$

11. Find the sum of the first 25 terms of the arithmetic series $3 + 7 + 11 + 15 + \cdots$

Step 1: Identify $a_1 = 3$, $d = 4$, $n = 25$.
Step 2: Apply the arithmetic series sum formula.

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

Step 3: Substitute.

$$S_{25} = \frac{25}{2}\left[2(3) + (24)(4)\right] = \frac{25}{2}\left[6 + 96\right] = \frac{25}{2} \times 102 = 25 \times 51 = 1275$$

$$\boxed{S_{25} = 1275}$$

12. Find the sum of the geometric series $5 + 10 + 20 + 40 + 80$.

Step 1: Identify $a_1 = 5$, $r = 2$, $n = 5$.
Step 2: Apply the geometric series sum formula.

$$S_5 = \frac{a_1(r^n - 1)}{r - 1} = \frac{5(2^5 - 1)}{2 - 1} = \frac{5(32 - 1)}{1} = 5 \times 31 = 155$$

$$\boxed{S_5 = 155}$$

13. Find the sum of the infinite geometric series $12 + 6 + 3 + 1.5 + \cdots$

Step 1: Identify $a_1 = 12$ and $r = \frac{6}{12} = \frac{1}{2}$.
Step 2: Confirm convergence: $\left|\frac{1}{2}\right| < 1$. ✓
Step 3: Apply the infinite sum formula.

$$S_\infty = \frac{a_1}{1 - r} = \frac{12}{1 - \frac{1}{2}} = \frac{12}{\frac{1}{2}} = 24$$

$$\boxed{S_\infty = 24}$$

14. A ball bounces to 80% of its previous height, dropped from 5 m. Find the total distance.

Step 1: The ball drops 5 m initially.
Step 2: After each bounce, it rises and falls. The upward distances form an infinite geometric series with $a_1 = 5 \times 0.8 = 4$ and $r = 0.8$.

$$S_{\text{up}} = \frac{4}{1 - 0.8} = \frac{4}{0.2} = 20 \text{ m}$$

Step 3: By symmetry, the total downward bouncing distance also equals 20 m.

Step 4: Add all parts.

$$\text{Total} = 5 + 20 + 20 = 45 \text{ m}$$

$$\boxed{45 \text{ metres}}$$

15. Find the center and radius of $x^2 + y^2 - 6x + 4y - 3 = 0$.

Step 1: Group and rearrange.

$$(x^2 - 6x) + (y^2 + 4y) = 3$$

Step 2: Complete the square for x: $\left(\frac{-6}{2}\right)^2 = 9$.

$$(x^2 - 6x + 9) + (y^2 + 4y) = 3 + 9$$

Step 3: Complete the square for y: $\left(\frac{4}{2}\right)^2 = 4$.

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 3 + 9 + 4$$

Step 4: Write in standard form.

$$(x - 3)^2 + (y + 2)^2 = 16$$

Center $(3, -2)$ and $r = \sqrt{16} = 4$.

$$\boxed{\text{Center } (3,\ -2), \quad r = 4}$$

16. For the parabola $y = -2(x - 1)^2 + 3$, identify the vertex and direction.

Step 1: The equation is in vertex form $y = a(x - h)^2 + k$. Read off $h = 1$, $k = 3$: vertex is $(1, 3)$.

Step 2: The leading coefficient is $a = -2 < 0$, so the parabola opens **downward**.

$$\boxed{\text{Vertex } (1,\ 3),\ \text{opens downward}}$$

17. For the ellipse $\frac{x^2}{49} + \frac{y^2}{25} = 1$, find the lengths of the major and minor axes.

Step 1: Identify $a^2 = 49$ and $b^2 = 25$ (since $49 > 25$).

$$a = 7 \qquad b = 5$$

Step 2: Compute the axis lengths.

$$\text{Major axis} = 2a = 14 \qquad \text{Minor axis} = 2b = 10$$

$$\boxed{\text{Major axis} = 14, \quad \text{Minor axis} = 10}$$

18. For the hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$, what are the vertices?

Step 1: The positive x^2 term means the transverse axis is along the x-axis.
Step 2: Identify $a^2 = 25$, so $a = 5$.
Step 3: The vertices are at $(\pm a, 0)$.

$$\boxed{\text{Vertices: } (\pm 5,\ 0)}$$

19. Evaluate $2^{7/2}$.

Step 1: Rewrite the exponent as a sum: $\dfrac{7}{2} = 3 + \dfrac{1}{2}$.

$$2^{7/2} = 2^3 \cdot 2^{1/2}$$

Step 2: Evaluate each factor.

$$2^3 = 8 \qquad 2^{1/2} = \sqrt{2}$$

Step 3: Multiply.

$$2^{7/2} = 8\sqrt{2}$$

$$\boxed{2^{7/2} = 8\sqrt{2}}$$

20. Describe the transformation of $g(x) = 2^{x+2} - 3$ compared to $f(x) = 2^x$.

Explanation: Apply the standard transformation rules:

- Replacing x with $(x + 2)$ in the exponent shifts the graph **left by 2 units**. (Adding inside shifts left — the opposite of what feels intuitive.)
- Subtracting 3 outside the exponential shifts the graph **down by 3 units**.

$$\boxed{\text{Shifted left 2 units and down 3 units}}$$

21. Convert $\log_4(64) = x$ to exponential form and solve.

Step 1: Convert using the rule $\log_b(x) = y \Leftrightarrow b^y = x$.

$$\log_4(64) = x \quad \implies \quad 4^x = 64$$

Step 2: Express 64 as a power of 4.

$$64 = 4^3$$

Step 3: Match exponents.

$$x = 3$$

$$\boxed{4^x = 64, \quad x = 3}$$

22. Expand $\log_3\left(\dfrac{9x^2}{y}\right)$.

Step 1: Apply the Quotient Rule.

$$\log_3\left(\frac{9x^2}{y}\right) = \log_3(9x^2) - \log_3(y)$$

Step 2: Apply the Product Rule to $\log_3(9x^2)$.

$$= \log_3(9) + \log_3(x^2) - \log_3(y)$$

Step 3: Apply the Power Rule to $\log_3(x^2)$, and evaluate $\log_3(9) = 2$ (since $3^2 = 9$).

$$= 2 + 2\log_3(x) - \log_3(y)$$

$$\boxed{2 + 2\log_3(x) - \log_3(y)}$$

23. Solve $\log_5(2x - 1) = 2$.

Step 1: Convert to exponential form.

$$5^2 = 2x - 1$$

Step 2: Evaluate the left side.

$$25 = 2x - 1$$

Step 3: Add 1 to both sides.

$$26 = 2x$$

Step 4: Divide by 2.

$$x = 13$$

Step 5: Check the argument is positive: $2(13) - 1 = 25 > 0$. ✓

$$\boxed{x = 13}$$

24. A radioactive substance decays by $A(t) = 100(0.5)^{t/5}$. How much remains after 10 years?

Step 1: Substitute $t = 10$.

$$A(10) = 100(0.5)^{10/5} = 100(0.5)^2$$

Step 2: Evaluate.

$$A(10) = 100 \times 0.25 = 25$$

$$\boxed{\text{25 grams}}$$

25. What is the domain of $f(x) = \sqrt{4 - x^2}$?

Step 1: The radicand must be ≥ 0.

$$4 - x^2 \geq 0$$

Step 2: Rearrange.

$$x^2 \leq 4$$

Step 3: Take the square root of both sides (remember to include both signs).

$$|x| \leq 2 \qquad \Rightarrow \qquad -2 \leq x \leq 2$$

$$\boxed{-2 \leq x \leq 2}$$

26. If $f(x) = x^2 + 1$ and $g(x) = 3x - 2$, find $(g \circ f)(2)$.

Step 1: Work from the inside out. Evaluate $f(2)$ first.

$$f(2) = (2)^2 + 1 = 5$$

Step 2: Substitute the result into g.

$$g(f(2)) = g(5) = 3(5) - 2 = 15 - 2 = 13$$

$$\boxed{(g \circ f)(2) = 13}$$

27. If $f(x) = \dfrac{4x+1}{3}$, find $f^{-1}(3)$.

Step 1: Find $f^{-1}(x)$ by swapping x and y and solving.

$$y = \frac{4x+1}{3} \quad \Rightarrow \quad x = \frac{4y+1}{3} \quad \Rightarrow \quad 3x = 4y+1 \quad \Rightarrow \quad y = \frac{3x-1}{4}$$

Step 2: Evaluate at $x = 3$.

$$f^{-1}(3) = \frac{3(3)-1}{4} = \frac{8}{4} = 2$$

$$\boxed{f^{-1}(3) = 2}$$

28. Simplify $\sqrt[4]{16x^4y^8}$.

Step 1: Separate the fourth root across each factor.

$$\sqrt[4]{16x^4y^8} = \sqrt[4]{16} \cdot \sqrt[4]{x^4} \cdot \sqrt[4]{y^8}$$

Step 2: Evaluate each factor using the rule $\sqrt[4]{a^n} = a^{n/4}$.

$$\sqrt[4]{16} = 16^{1/4} = (2^4)^{1/4} = 2 \qquad \sqrt[4]{x^4} = x \qquad \sqrt[4]{y^8} = y^{8/4} = y^2$$

Step 3: Multiply the results.

$$= 2 \cdot x \cdot y^2 = 2xy^2$$

$$\boxed{2xy^2}$$

29. Factor $x^3 - 3x^2 - 4x + 12$ by grouping.

Step 1: Split into two pairs.

$$(x^3 - 3x^2) + (-4x + 12)$$

Step 2: Factor the GCF from each pair.

$$x^2(x - 3) - 4(x - 3)$$

Step 3: Factor out the common binomial.

$$= (x - 3)(x^2 - 4)$$

Step 4: Factor $x^2 - 4$ as a difference of squares.

$$= (x - 3)(x - 2)(x + 2)$$

$$\boxed{(x - 3)(x - 2)(x + 2)}$$

30. Use synthetic division to divide $2x^3 + 3x^2 - 8x - 12$ by $(x + 2)$.

Step 1: The divisor is $(x + 2)$, so the synthetic division value is $c = -2$. Write the coefficients of the dividend.

$$\begin{array}{r|rrrr} -2 & 2 & 3 & -8 & -12 \\ & & -4 & 2 & 12 \\ \hline & 2 & -1 & -6 & 0 \end{array}$$

Step 2: Carry down 2. Multiply $-2 \times 2 = -4$; add to 3 to get -1. Multiply $-2 \times (-1) = 2$; add to -8 to get -6. Multiply $-2 \times (-6) = 12$; add to -12 to get 0.

Step 3: The bottom row gives the quotient coefficients and remainder. The quotient is $2x^2 - x - 6$ and the remainder is 0.

$$\boxed{\text{Quotient: } 2x^2 - x - 6, \quad \text{Remainder: } 0}$$

31. For $f(x) = (x + 1)^3(x - 2)^2(x - 4)$, what is the multiplicity of the zero at $x = -1$?

Explanation: The zero $x = -1$ comes from the factor $(x + 1)$. The exponent on that factor is 3, which is the multiplicity. At a zero of odd multiplicity, the graph crosses through the x-axis.

$$\boxed{\text{Multiplicity } 3}$$

32. For $f(x) = 3x^5 - 2x^3 + x$, describe the end behaviour.

Step 1: The end behaviour is determined by the leading term $3x^5$.

Step 2: The degree is 5 (odd), so the two ends go in opposite directions. The leading coefficient is $+3$ (positive), so the right end rises.

- As $x \to +\infty$: $f(x) \to +\infty$ (up on the right)
- As $x \to -\infty$: $f(x) \to -\infty$ (down on the left)

$$\boxed{\text{Down on the left, up on the right}}$$

33. The height of a Ferris wheel seat is $h(t) = 30 + 25\sin\left(\frac{\pi t}{15}\right)$. What is the maximum height?

Step 1: The maximum value of $\sin(\cdot)$ is 1. Substitute this to find the maximum height.

$$h_{\max} = 30 + 25(1) = 55 \text{ m}$$

$$\boxed{h_{\max} = 55 \text{ metres}}$$

34. You deposit \$100 at the end of each month into an account earning 0.5% monthly interest. What is the total value after 12 deposits?

Step 1: Use the future value of an ordinary annuity formula.

$$FV = PMT \cdot \frac{(1+i)^n - 1}{i}$$

Step 2: Identify $PMT = 100$, $i = 0.005$, $n = 12$.
Step 3: Substitute.

$$FV = 100 \cdot \frac{(1.005)^{12} - 1}{0.005}$$

Step 4: Evaluate $(1.005)^{12} \approx 1.06168$.

$$FV = 100 \cdot \frac{0.06168}{0.005} = 100 \times 12.3356 \approx 1233.56$$

$$\boxed{\$1{,}233.56}$$

35. If $\log_2(x) = 3 + \log_2(4)$, what is x?

Step 1: Rewrite the constant 3 as a logarithm. Since $2^3 = 8$, we have $3 = \log_2(8)$.

$$\log_2(x) = \log_2(8) + \log_2(4)$$

Step 2: Apply the Product Rule to the right side.

$$\log_2(x) = \log_2(8 \times 4) = \log_2(32)$$

Step 3: Since the logarithms are equal and have the same base, the arguments must be equal.

$$\boxed{x = 32}$$

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