Polynomial Functions Quiz
Answer Key
ApexMath

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1. Find the remainder when $P(x) = x^3 - 2x^2 + 3x - 4$ is divided by $(x - 2)$.

Step 1: Use the Remainder Theorem. When a polynomial $P(x)$ is divided by $(x - c)$, the remainder equals $P(c)$. Here $c = 2$.
Step 2: Evaluate $P(2)$.

$$P(2) = (2)^3 - 2(2)^2 + 3(2) - 4 = 8 - 8 + 6 - 4 = 2$$

$$\boxed{\text{Remainder} = 2}$$

2. Which of the following is a factor of $P(x) = x^3 + 4x^2 + x - 6$?

Step 1: Use the Factor Theorem. $(x - c)$ is a factor of $P(x)$ if and only if $P(c) = 0$.
Step 2: Test $x = 1$ by evaluating $P(1)$.

$$P(1) = (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0$$

Since $P(1) = 0$, the Factor Theorem confirms $(x - 1)$ is a factor.

$$\boxed{(x-1) \text{ is a factor}}$$

3. What is the quotient when $x^3 - 7x + 6$ is divided by $(x - 1)$?

Step 1: Note that the x^2 term is missing, so write the dividend with a $0x^2$ placeholder: $x^3 + 0x^2 - 7x + 6$. Perform polynomial long division.
Step 2: Divide x^3 by x to get x^2. Multiply and subtract.

$$x^3 + 0x^2 - 7x + 6 - x^2(x - 1) = x^3 + 0x^2 - 7x + 6 - (x^3 - x^2) = x^2 - 7x + 6$$

Step 3: Divide x^2 by x to get x. Multiply and subtract.

$$x^2 - 7x + 6 - x(x - 1) = x^2 - 7x + 6 - (x^2 - x) = -6x + 6$$

Step 4: Divide $-6x$ by x to get -6. Multiply and subtract.

$$-6x + 6 - (-6)(x - 1) = -6x + 6 - (-6x + 6) = 0$$

The remainder is 0, confirming $(x - 1)$ is a factor.

$$\boxed{\text{Quotient} = x^2 + x - 6}$$

4. List all possible rational zeros of $P(x) = 2x^3 + 3x^2 - 8x + 3$.

Step 1: Apply the Rational Zero Theorem. Every possible rational zero has the form:

$$\pm\frac{p}{q}$$

where p is a factor of the constant term and q is a factor of the leading coefficient.

Step 2: Identify the factors.

- Factors of the constant term 3: $\pm 1,\ \pm 3$
- Factors of the leading coefficient 2: $\pm 1,\ \pm 2$

Step 3: Form all combinations $\frac{p}{q}$ and remove duplicates.

$$\pm 1, \quad \pm 3, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}$$

$$\boxed{\pm 1, \quad \pm 3, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}}$$

5. Describe the end behaviour of $f(x) = -3x^5 + 2x^2 - 1$.

Step 1: End behaviour is determined entirely by the leading term $-3x^5$.
Step 2: Apply the end behaviour rules:

- The degree is 5 (odd), so the two ends of the graph go in *opposite* directions.
- The leading coefficient is -3 (negative), so the right side falls.

Step 3: Conclude: as $x \to +\infty$, $f(x) \to -\infty$ (down on the right); as $x \to -\infty$, $f(x) \to +\infty$ (up on the left).

$$\boxed{\text{Up on the left, down on the right}}$$

6. For $f(x) = (x-2)^3(x+1)^2$, what happens at the zero $x = -1$?

Explanation: The factor $(x+1)$ has exponent 2, so $x = -1$ is a zero of **even multiplicity**. At a zero of even multiplicity, the graph touches the x-axis and turns back around without crossing it — this is the behaviour of a parabola at its vertex.

By contrast, the zero at $x = 2$ has odd multiplicity (3), so the graph crosses through at that point.

$$\boxed{\text{The graph touches the } x\text{-axis and turns around at } x = -1}$$

7. Factor $x^3 - 8$ completely.

Step 1: Recognize this as a difference of cubes. The pattern is:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Step 2: Identify $a = x$ and $b = 2$ (since $2^3 = 8$).
Step 3: Substitute into the formula.

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

Step 4: Confirm the trinomial $x^2 + 2x + 4$ does not factor further over the reals (its discriminant is $4 - 16 = -12 < 0$).

$$\boxed{(x - 2)(x^2 + 2x + 4)}$$

8. Find all real zeros of $f(x) = x^3 - x^2 - 4x + 4$.

Step 1: Try factoring by grouping. Split the polynomial into two pairs.

$$(x^3 - x^2) + (-4x + 4)$$

Step 2: Factor the GCF from each pair.

$$x^2(x - 1) - 4(x - 1)$$

Step 3: Factor out the common binomial $(x - 1)$.

$$= (x - 1)(x^2 - 4)$$

Step 4: Factor $x^2 - 4$ as a difference of squares.

$$= (x - 1)(x - 2)(x + 2)$$

Step 5: Set each factor equal to zero.

$$x = 1, \qquad x = 2, \qquad x = -2$$

$$\boxed{x = 1, \quad x = 2, \quad x = -2}$$

9. If $P(x)$ is divided by $(x - a)$ and the remainder is 0, then:

Explanation: This is the Factor Theorem, a direct consequence of the Remainder Theorem. If dividing $P(x)$ by $(x - a)$ yields a remainder of 0, then:

- $(x - a)$ divides $P(x)$ evenly, meaning it is a **factor** of $P(x)$.
- $P(a) = 0$, meaning $x = a$ is a **root** (zero) of $P(x)$.

$$\boxed{x = a \text{ is a root (zero) of } P(x)}$$

10. A cubic polynomial with real coefficients has roots 2 and $1+i$. What is the third root?

Explanation: The Complex Conjugate Root Theorem states: if a polynomial has *real* coefficients, then complex roots always appear in conjugate pairs. If $1+i$ is a root, then its conjugate $1-i$ must also be a root.

Since the polynomial is cubic (degree 3), it has exactly 3 roots: 2, $1+i$, and $1-i$.

$$\boxed{1-i}$$

11. A polynomial of degree 4 can have at most how many x-intercepts?

Explanation: The x-intercepts of a polynomial correspond to its real zeros. A degree-4 polynomial has exactly 4 roots (counted with multiplicity, including complex roots). In the best case, all 4 roots are real and distinct, giving 4 separate x-intercepts. Therefore the maximum is 4.

$$\boxed{\text{4 x-intercepts}}$$

12. A polynomial of degree 6 can have at most how many turning points?

Explanation: A turning point is a local maximum or minimum — a place where the graph changes direction. A polynomial of degree n can have at most $n-1$ turning points.

$$\text{Max turning points} = n - 1 = 6 - 1 = 5$$

$$\boxed{\text{5 turning points}}$$

13. If $P(1) = -5$ and $P(2) = 3$, what can we conclude about the zeros of $P(x)$?

Explanation: By the Intermediate Value Theorem (IVT): if $P(x)$ is a polynomial (continuous everywhere), and $P(1) = -5 < 0$ while $P(2) = 3 > 0$, then P must change sign somewhere between $x = 1$ and $x = 2$. Since polynomials are continuous, the function must pass through zero at least once in that interval.

$$\boxed{\text{There is at least one zero between } x = 1 \text{ and } x = 2}$$

14. If the leading coefficient is positive and the degree is even, the graph:

Explanation: For even degree polynomials, both ends of the graph point in the same direction. When the leading coefficient is positive, large values of $|x|$ make the leading term large and positive, so both ends rise upward.
Mnemonic: Think of $y = x^2$ — positive, even degree, rises on both sides.

$$\boxed{\text{Rises to the left and to the right}}$$

15. Divide $2x^3 - 5x^2 - 4x + 3$ by $(x - 3)$.

Step 1: Divide $2x^3$ by x to get $2x^2$. Multiply and subtract.

$$2x^3 - 5x^2 - 4x + 3 - 2x^2(x - 3) = 2x^3 - 5x^2 - 4x + 3 - (2x^3 - 6x^2) = x^2 - 4x + 3$$

Step 2: Divide x^2 by x to get x. Multiply and subtract.

$$x^2 - 4x + 3 - x(x - 3) = x^2 - 4x + 3 - (x^2 - 3x) = -x + 3$$

Step 3: Divide $-x$ by x to get -1. Multiply and subtract.

$$-x + 3 - (-1)(x - 3) = -x + 3 - (-x + 3) = 0$$

The remainder is 0, so $(x - 3)$ is a factor.

$$\boxed{\text{Quotient} = 2x^2 + x - 1}$$

16. Factor $2x^3 + 250$ completely.

Step 1: Factor out the GCF of 2.

$$2x^3 + 250 = 2(x^3 + 125)$$

Step 2: Recognize $x^3 + 125$ as a sum of cubes. The pattern is:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Step 3: Identify $a = x$ and $b = 5$ (since $5^3 = 125$). Substitute.

$$x^3 + 125 = (x + 5)(x^2 - 5x + 25)$$

Step 4: Write the fully factored form.

$$2x^3 + 250 = 2(x + 5)(x^2 - 5x + 25)$$

$$\boxed{2(x + 5)(x^2 - 5x + 25)}$$

17. If a polynomial has zeros at $x = 0$ (multiplicity 2) and $x = 5$ (multiplicity 1), what is the minimum degree?

Explanation: The degree of a polynomial is at least the sum of all the multiplicities of its zeros. Add up the given multiplicities.

$$\text{Min degree} = 2 + 1 = 3$$

The simplest such polynomial is $f(x) = x^2(x - 5) = x^3 - 5x^2$.

$$\boxed{\text{Minimum degree} = 3}$$

18. Find a rational zero of $P(x) = x^3 - 3x^2 - x + 3$.

Step 1: By the Rational Zero Theorem, possible rational zeros are ± 1 and ± 3.
Step 2: Test $x = 3$.

$$P(3) = (3)^3 - 3(3)^2 - (3) + 3 = 27 - 27 - 3 + 3 = 0$$

Since $P(3) = 0$, the value $x = 3$ is a rational zero.

$$\boxed{x = 3}$$

19. If $P(x) = x^4 - kx^2 + 4$ and $P(2) = 0$, find k.

Step 1: Substitute $x = 2$ into $P(x)$ and set equal to 0.

$$P(2) = (2)^4 - k(2)^2 + 4 = 0$$

Step 2: Evaluate the powers.

$$16 - 4k + 4 = 0$$

Step 3: Combine constants.

$$20 - 4k = 0$$

Step 4: Solve for k.

$$4k = 20 \qquad \Rightarrow \qquad k = 5$$

$$\boxed{k = 5}$$

20. A polynomial with only even powers of x is:

Explanation: A function is called **even** if $f(-x) = f(x)$ for all x. This means replacing x with $-x$ leaves the function unchanged — the graph is symmetric about the y-axis.

For a polynomial with only even powers (e.g., $ax^4 + bx^2 + c$), every term contains x raised to an even power. Since $(-x)^{2n} = x^{2n}$, substituting $-x$ does not change any term.

$$\boxed{\text{An even function (symmetric about the } y\text{-axis)}}$$

21. If the graph of $P(x)$ crosses the x-axis at $x = c$, the multiplicity of $(x - c)$ must be:

Explanation: The behaviour of a polynomial at a zero depends on the zero's multiplicity:

- **Odd multiplicity**: the graph *crosses* through the x-axis at $x = c$. The function changes sign at this point.
- **Even multiplicity**: the graph *touches* the x-axis and bounces back without crossing.

Since the graph is stated to *cross* at $x = c$, the multiplicity must be odd (e.g., 1, 3, 5, …).

$$\boxed{\text{Odd}}$$

22. Divide $x^2 + 5x + 6$ by $(x + 2)$.

Step 1: Divide x^2 by x to get x. Multiply and subtract.

$$x^2 + 5x + 6 - x(x + 2) = x^2 + 5x + 6 - (x^2 + 2x) = 3x + 6$$

Step 2: Divide $3x$ by x to get 3. Multiply and subtract.

$$3x + 6 - 3(x + 2) = 3x + 6 - (3x + 6) = 0$$

The remainder is 0.

$$\boxed{\text{Quotient} = x + 3}$$

23. Find a cubic polynomial with zeros -1, 1, and 2.

Step 1: If a polynomial has zeros at $x = -1$, $x = 1$, and $x = 2$, then its factors are $(x + 1)$, $(x - 1)$, and $(x - 2)$.

Step 2: Write the polynomial as the product of its factors (using leading coefficient 1).

$$P(x) = (x + 1)(x - 1)(x - 2)$$

Step 3: Expand step by step. First multiply $(x+1)(x-1)$ using the difference of squares.

$$(x + 1)(x - 1) = x^2 - 1$$

Step 4: Multiply the result by $(x-2)$.

$$(x^2-1)(x-2) = x^3 - 2x^2 - x + 2$$

$$\boxed{P(x) = x^3 - 2x^2 - x + 2}$$

24. Find the y-intercept of $P(x) = 4x^3 - 5x^2 + 7x - 9$.

Step 1: The y-intercept occurs where $x = 0$. Substitute $x = 0$.

$$P(0) = 4(0)^3 - 5(0)^2 + 7(0) - 9 = 0 - 0 + 0 - 9 = -9$$

Note: For any polynomial, the y-intercept is always the constant term. No calculation is required once you see this pattern.

$$\boxed{y\text{-intercept} = (0,\ -9)}$$

25. Which function is a polynomial?

Explanation: A polynomial function consists only of non-negative integer powers of x with real coefficients. Common expressions that disqualify a function from being a polynomial include:

- Variables in the denominator (e.g., $\frac{1}{x}$)
- Variables under a radical (e.g., $\sqrt{x}$)
- Variables as exponents (e.g., 2^x)
- Absolute value of a variable (e.g., $|x|$)

The function $f(x) = 3x^2 - 5x + 2$ has only non-negative integer powers of x (x^2 and x^1) with real coefficients. It satisfies all requirements.

$$\boxed{f(x) = 3x^2 - 5x + 2}$$

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