Antiderivatives
ApexMath

Notes

What is an Antiderivative?

So far in calculus we have spent a lot of time asking: *given a function $f(x)$, what is its derivative?* An antiderivative flips that question around: *given a function $f(x)$, what function has $f(x)$ as its derivative?*

Formally, $F(x)$ is an **antiderivative** of $f(x)$ if:

$$F'(x) = f(x)$$

For example, $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, because differentiating x^3 gives $3x^2$.

Why $+C$? The Family of Antiderivatives

Here is something important: antiderivatives are not unique. Both x^3 and $x^3 + 7$ and $x^3 - 15$ are all antiderivatives of $3x^2$, because constants disappear when you differentiate.

This means every function has infinitely many antiderivatives, one for each possible constant. We capture all of them at once by writing $+C$, where C represents any real number constant. This collection of all antiderivatives is called the **general antiderivative**:

$$F(x) + C$$

Geometrically, the $+C$ represents a whole family of curves that are vertical shifts of each other. They all have the same shape and slope at every x-value, just shifted up or down.

The Reverse Power Rule

The most important antiderivative rule is the reverse of the power rule for derivatives. For any $n \neq -1$:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

In words: add one to the exponent, then divide by the new exponent.

Why does $n = -1$ not work? If $n = -1$, the formula would require dividing by zero ($n + 1 = 0$), which is undefined. The special case $n = -1$ has its own rule:

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

The absolute value is important here because ln is only defined for positive numbers, but $\frac{1}{x}$ exists for negative x as well.

Complete Antiderivative Reference Table

Function $f(x)$	**Antiderivative** $F(x)+C$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}+C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
e^{kx}	$\dfrac{1}{k}e^{kx} + C$
a^x	$\dfrac{a^x}{\ln a} + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\csc^2(x)$	$-\cot(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
$\csc(x)\cot(x)$	$-\csc(x) + C$

You can always verify any antiderivative by differentiating your answer. If $\dfrac{d}{dx}[F(x)] = f(x)$, the antiderivative is correct.

Rewriting Before Integrating

The reverse power rule only works when the function is written as a power of x. Before integrating, you often need to rewrite the function first. Common rewrites include:

- $\sqrt{x} = x^{1/2}$
- $\sqrt[3]{x} = x^{1/3}$
- $\dfrac{1}{x^n} = x^{-n}$
- $\dfrac{1}{\sqrt{x}} = x^{-1/2}$
- Expand products: $x(x^2+3) = x^3 + 3x$ before integrating

Initial Value Problems

An **initial value problem** gives you $f'(x)$ and one specific value of f, such as $f(0) = 4$. The goal is to find the particular antiderivative that satisfies that condition — in other words, to find the exact value of C.

The process is:

Step 1: Find the general antiderivative $F(x) + C$.
Step 2: Substitute the given point to create an equation for C.
Step 3: Solve for C.
Step 4: Write the final answer with the specific value of C substituted in.

Pro Tip

- Before applying the reverse power rule, always rewrite roots and fractions as powers of x with exponents. Trying to integrate $\sqrt{x}$ directly without converting to $x^{1/2}$ first is a very common mistake.
- After finding an antiderivative, always differentiate your answer to verify it. This takes ten seconds and catches errors immediately.
- The reverse power rule says: add one to the exponent, *then* divide by that new exponent. Students often divide by the original exponent by mistake. The dividing happens *after* adding one.
- For initial value problems, find the general antiderivative *completely* (including $+C$) before substituting the initial condition. Substituting too early causes errors.
- When you have a second-order initial value problem (given f'', find f), integrate *twice*. Each integration introduces a new constant — call them C_1 and C_2 — and you need two conditions to find both.

Examples

Example 1 (Reverse Power Rule): Find the general antiderivative of $f(x) = 6x^4 - 3x^2 + 5x - 8$.

Step 1: Apply the reverse power rule term by term.

For each term ax^n, the antiderivative is $a \cdot \dfrac{x^{n+1}}{n+1}$.

$$\int (6x^4 - 3x^2 + 5x - 8)\, dx$$

Step 2: Integrate each term.

$$= 6 \cdot \frac{x^5}{5} - 3 \cdot \frac{x^3}{3} + 5 \cdot \frac{x^2}{2} - 8x + C$$

Step 3: Simplify.

$$= \frac{6x^5}{5} - x^3 + \frac{5x^2}{2} - 8x + C$$

$$\boxed{F(x) = \frac{6x^5}{5} - x^3 + \frac{5x^2}{2} - 8x + C}$$

Verification: Differentiating $F(x)$ gives $6x^4 - 3x^2 + 5x - 8 = f(x)$. ✓

Example 2 (Rewriting First): Find $\displaystyle\int \left(\sqrt{x} + \frac{3}{x^2} - \frac{1}{\sqrt{x}}\right) dx$.

Step 1: Rewrite every term as a power of x.

$$\sqrt{x} = x^{1/2}, \qquad \frac{3}{x^2} = 3x^{-2}, \qquad \frac{1}{\sqrt{x}} = x^{-1/2}$$

$$\int \left(x^{1/2} + 3x^{-2} - x^{-1/2}\right) dx$$

Step 2: Apply the reverse power rule to each term.

$$= \frac{x^{3/2}}{3/2} + 3 \cdot \frac{x^{-1}}{-1} - \frac{x^{1/2}}{1/2} + C$$

Step 3: Simplify the fractions.

Recall that dividing by $\frac{3}{2}$ is the same as multiplying by $\frac{2}{3}$, and dividing by $\frac{1}{2}$ is the same as multiplying by 2.

$$= \frac{2}{3}x^{3/2} - \frac{3}{x} - 2x^{1/2} + C$$

$$\boxed{F(x) = \frac{2}{3}x^{3/2} - \frac{3}{x} - 2\sqrt{x} + C}$$

Example 3 (Expand First): Find $\displaystyle\int x^2(x-4)\, dx$.

Step 1: Expand the product before integrating.

$$x^2(x-4) = x^3 - 4x^2$$

Step 2: Integrate term by term.

$$\int \left(x^3 - 4x^2\right) dx = \frac{x^4}{4} - \frac{4x^3}{3} + C$$

$$\boxed{F(x) = \frac{x^4}{4} - \frac{4x^3}{3} + C}$$

Example 4 (Trig and Exponential): Find $\int (3\sin(x) - 2e^x + \cos(x))\,dx$.

Step 1: Apply the antiderivative rules to each term separately.

$$\int 3\sin(x)\,dx = 3(-\cos(x)) = -3\cos(x)$$

$$\int -2e^x\,dx = -2e^x$$

$$\int \cos(x)\,dx = \sin(x)$$

Step 2: Combine and add $+C$.

$$\boxed{F(x) = -3\cos(x) - 2e^x + \sin(x) + C}$$

Example 5 (Initial Value Problem): Given $f'(x) = 4x^3 - 6x$ and $f(1) = 2$, find $f(x)$.

Step 1: Find the general antiderivative of $f'(x)$.

$$f(x) = \int (4x^3 - 6x)\,dx = x^4 - 3x^2 + C$$

Step 2: Substitute the initial condition $f(1) = 2$.

$$f(1) = (1)^4 - 3(1)^2 + C = 2$$

$$1 - 3 + C = 2$$

$$-2 + C = 2$$

Step 3: Solve for C.

$$C = 4$$

Step 4: Write the final answer.

$$\boxed{f(x) = x^4 - 3x^2 + 4}$$

Example 6 (Second-Order Initial Value Problem): Given $f''(x) = 6x+2$, $f'(0) = 1$, and $f(0) = 3$, find $f(x)$.

Step 1: Integrate $f''(x)$ once to find $f'(x)$.

$$f'(x) = \int (6x + 2)\, dx = 3x^2 + 2x + C_1$$

Step 2: Use $f'(0) = 1$ to find C_1.

$$f'(0) = 3(0)^2 + 2(0) + C_1 = 1 \implies C_1 = 1$$

So $f'(x) = 3x^2 + 2x + 1$.

Step 3: Integrate $f'(x)$ to find $f(x)$.

$$f(x) = \int (3x^2 + 2x + 1)\, dx = x^3 + x^2 + x + C_2$$

Step 4: Use $f(0) = 3$ to find C_2.

$$f(0) = (0)^3 + (0)^2 + (0) + C_2 = 3 \implies C_2 = 3$$

Step 5: Write the final answer.

$$\boxed{f(x) = x^3 + x^2 + x + 3}$$

Common Mistakes

- **Dividing by the original exponent instead of the new one:** The reverse power rule says add one first, *then* divide by the result. For x^4, you get $\dfrac{x^5}{5}$, not $\dfrac{x^5}{4}$.
- **Forgetting $+C$:** Every indefinite antiderivative must include $+C$. Dropping it means your answer describes only one function instead of the entire family.
- **Not rewriting roots and fractions first:** The reverse power rule only works on expressions written as x^n. You cannot directly integrate $\sqrt{x}$ or $\frac{1}{x^2}$ without first rewriting them as $x^{1/2}$ and x^{-2}.
- **Trying to use the power rule for $\frac{1}{x}$:** Plugging $n = -1$ into the reverse power rule gives $\frac{x^0}{0}$, which is undefined. The antiderivative of $\frac{1}{x}$ is $\ln|x| + C$, a completely separate rule.
- **Using the wrong sign for trig antiderivatives:** The antiderivative of $\sin(x)$ is $-\cos(x)$, not $+\cos(x)$. Forgetting the negative sign is one of the most common errors in this topic.
- **Forgetting to introduce a new constant for each integration in second-order problems:** Every time you integrate, a new C appears. If you integrate twice, you need C_1 and C_2, and you need two initial conditions to find both.

Worksheet

1. Find the general antiderivative of $f(x) = 8x^3 - 5x + 2$.

2. Find $\displaystyle\int \left(x^6 - \frac{1}{2}x^4 + 3\right) dx$.

3. Find $\displaystyle\int \left(\frac{4}{x^3} + \sqrt[3]{x}\right) dx$.

   (Rewrite as powers of x first.)

4. Find $\displaystyle\int \frac{x^3 - 5x}{\sqrt{x}}\, dx$.

   (Divide each term by $\sqrt{x} = x^{1/2}$ first.)

5. Find $\int x(3x+1)^2\,dx$.

(Expand the product completely first.)

6. Find $\int (4\cos(x) - \sec^2(x))\,dx$.

7. Find $\int \left(7e^x - \frac{3}{x} + 2\sin(x)\right)\,dx$.

8. Given $f'(x) = 3x^2 - 4x + 1$ and $f(2) = 5$, find $f(x)$.

9. Given $f'(x) = \cos(x)$ and $f(\pi) = 0$, find $f(x)$.

10. (Challenge) Given $f''(x) = 12x - 6$, $f'(0) = 4$, and $f(1) = 2$, find $f(x)$.

11. (Challenge) A particle moves along a line with acceleration $a(t) = 6t - 2$. At time $t = 0$, the velocity is $v(0) = 3$ m/s and the position is $s(0) = -1$ m. Find the position function $s(t)$.

Answer Key

1. $f(x) = 8x^3 - 5x + 2$

Apply the reverse power rule to each term:

$$\int (8x^3 - 5x + 2)\,dx = 8 \cdot \frac{x^4}{4} - 5 \cdot \frac{x^2}{2} + 2x + C$$

$$\boxed{F(x) = 2x^4 - \frac{5x^2}{2} + 2x + C}$$

2. $\displaystyle\int \left(x^6 - \frac{1}{2}x^4 + 3\right) dx$

$$= \frac{x^7}{7} - \frac{1}{2} \cdot \frac{x^5}{5} + 3x + C$$

$$\boxed{F(x) = \frac{x^7}{7} - \frac{x^5}{10} + 3x + C}$$

3. $\displaystyle\int \left(\frac{4}{x^3} + \sqrt[3]{x}\right) dx$

Step 1: Rewrite.

$$\frac{4}{x^3} = 4x^{-3}, \qquad \sqrt[3]{x} = x^{1/3}$$

Step 2: Integrate.

$$\int (4x^{-3} + x^{1/3})\,dx = 4 \cdot \frac{x^{-2}}{-2} + \frac{x^{4/3}}{4/3} + C = -\frac{2}{x^2} + \frac{3}{4}x^{4/3} + C$$

$$\boxed{F(x) = -\frac{2}{x^2} + \frac{3}{4}x^{4/3} + C}$$

4. $\displaystyle\int \frac{x^3 - 5x}{\sqrt{x}}\,dx$

Step 1: Divide each term by $x^{1/2}$.

$$\frac{x^3}{x^{1/2}} - \frac{5x}{x^{1/2}} = x^{5/2} - 5x^{1/2}$$

Step 2: Integrate.

$$\int (x^{5/2} - 5x^{1/2})\,dx = \frac{x^{7/2}}{7/2} - 5 \cdot \frac{x^{3/2}}{3/2} + C = \frac{2}{7}x^{7/2} - \frac{10}{3}x^{3/2} + C$$

$$\boxed{F(x) = \frac{2}{7}x^{7/2} - \frac{10}{3}x^{3/2} + C}$$

5. $\int x(3x+1)^2\,dx$
Step 1: Expand $(3x+1)^2 = 9x^2 + 6x + 1$.
Step 2: Multiply by x:

$$x(9x^2 + 6x + 1) = 9x^3 + 6x^2 + x$$

Step 3: Integrate.

$$\int (9x^3 + 6x^2 + x)\,dx = \frac{9x^4}{4} + 2x^3 + \frac{x^2}{2} + C$$

$$\boxed{F(x) = \frac{9x^4}{4} + 2x^3 + \frac{x^2}{2} + C}$$

6. $\int (4\cos(x) - \sec^2(x))\,dx$
Apply antiderivative rules term by term:

$$\int 4\cos(x)\,dx = 4\sin(x), \qquad \int -\sec^2(x)\,dx = -\tan(x)$$

$$\boxed{F(x) = 4\sin(x) - \tan(x) + C}$$

7. $\int \left(7e^x - \frac{3}{x} + 2\sin(x)\right)dx$

$$\int 7e^x\,dx = 7e^x, \qquad \int -\frac{3}{x}\,dx = -3\ln|x|, \qquad \int 2\sin(x)\,dx = -2\cos(x)$$

$$\boxed{F(x) = 7e^x - 3\ln|x| - 2\cos(x) + C}$$

8. $f'(x) = 3x^2 - 4x + 1$, $f(2) = 5$.
Step 1: Find the general antiderivative.

$$f(x) = x^3 - 2x^2 + x + C$$

Step 2: Substitute $f(2) = 5$:

$$(2)^3 - 2(2)^2 + 2 + C = 5$$

$$8 - 8 + 2 + C = 5$$

$$2 + C = 5 \implies C = 3$$

$$\boxed{f(x) = x^3 - 2x^2 + x + 3}$$

9. $f'(x) = \cos(x)$, $f(\pi) = 0$.

Step 1: Find the general antiderivative.

$$f(x) = \sin(x) + C$$

Step 2: Substitute $f(\pi) = 0$:

$$\sin(\pi) + C = 0 \implies 0 + C = 0 \implies C = 0$$

$$\boxed{f(x) = \sin(x)}$$

10. (Challenge) $f''(x) = 12x - 6$, $f'(0) = 4$, $f(1) = 2$.
Step 1: Integrate $f''(x)$ to find $f'(x)$.

$$f'(x) = \int (12x - 6)\, dx = 6x^2 - 6x + C_1$$

Step 2: Use $f'(0) = 4$:

$$6(0)^2 - 6(0) + C_1 = 4 \implies C_1 = 4$$

So $f'(x) = 6x^2 - 6x + 4$.
Step 3: Integrate $f'(x)$ to find $f(x)$.

$$f(x) = \int (6x^2 - 6x + 4)\, dx = 2x^3 - 3x^2 + 4x + C_2$$

Step 4: Use $f(1) = 2$:

$$2(1)^3 - 3(1)^2 + 4(1) + C_2 = 2$$
$$2 - 3 + 4 + C_2 = 2$$
$$3 + C_2 = 2 \implies C_2 = -1$$
$$\boxed{f(x) = 2x^3 - 3x^2 + 4x - 1}$$

11. (Challenge) $a(t) = 6t - 2$, $v(0) = 3$, $s(0) = -1$.
Step 1: Integrate $a(t)$ to find $v(t)$.

$$v(t) = \int (6t - 2)\, dt = 3t^2 - 2t + C_1$$

Step 2: Use $v(0) = 3$:

$$3(0)^2 - 2(0) + C_1 = 3 \implies C_1 = 3$$

So $v(t) = 3t^2 - 2t + 3$.
Step 3: Integrate $v(t)$ to find $s(t)$.

$$s(t) = \int (3t^2 - 2t + 3)\, dt = t^3 - t^2 + 3t + C_2$$

Step 4: Use $s(0) = -1$:

$$(0)^3 - (0)^2 + 3(0) + C_2 = -1 \implies C_2 = -1$$

$$\boxed{s(t) = t^3 - t^2 + 3t - 1}$$

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