Approximating Solutions Using Euler's Method
ApexMath

AP Calculus BC Only

Note for Students: This topic is tested on the **AP Calculus BC** exam only. If you are in AP Calculus AB, you do not need to learn this material. BC students: this is a high-value topic that appears regularly on both the multiple-choice and free-response sections.

Notes

The Big Idea: Approximating When We Cannot Solve Exactly

Many differential equations cannot be solved with separation of variables or any other algebraic technique. Yet we often still need to know what the solution looks like.

Euler's method is a numerical technique that approximates the solution $y(x)$ by following the slope field one small step at a time. At each step, we move forward by a small horizontal distance h (called the **step size**) and use the current slope to estimate where the curve goes next.

The idea comes directly from the slope field: at any point (x_n, y_n), the differential equation tells us the slope $f(x_n, y_n)$. If we take one tiny step of width h to the right, a straight-line approximation brings us to:

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n,\, y_n)}$$

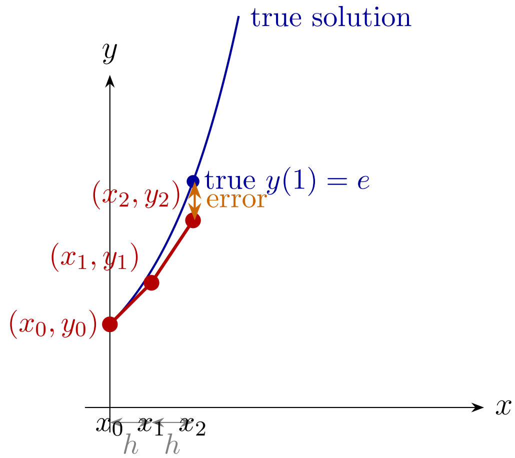
where $f(x, y) = \dfrac{dy}{dx}$ is the right-hand side of the differential equation. Repeating this step-by-step process builds up an approximation of the solution curve.

The Step-by-Step Process

1. **Identify:** $f(x, y)$, the initial condition (x_0, y_0), the step size h, and the target x-value.
2. **Compute the slope** at the current point: $m_n = f(x_n, y_n)$.
3. **Step forward:** $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \cdot m_n$.
4. **Repeat** from Step 2 using the new point (x_{n+1}, y_{n+1}).

It helps to organize each step in a table.

Visualizing Euler's Method



Each straight-line segment is tangent to the curve at its starting point, but drifts away from the true curve before the next correction. The smaller h is, the less drift accumulates at each step, and the more accurate the final estimate.

Key Properties of Euler's Method

- Euler's method always produces an **approximation**, never the exact value (unless the function happens to be linear).
- **Smaller step size** $\Rightarrow$ **more accurate** approximation, but more steps to compute.
- If the true solution is **concave up** at a point, the Euler approximation will be an **underestimate** there (the tangent line lies below the curve).
- If the true solution is **concave down**, the Euler approximation will be an **overestimate**.

Pro Tip

- **Always organize your work in a table.** Each row should have columns for x_n, y_n, $f(x_n, y_n)$, and $y_{n+1} = y_n + h \cdot f(x_n, y_n)$. This prevents cascading errors where one wrong step ruins every subsequent step.
- **The number of steps equals** $\dfrac{x_{\textbf{target}} - x_0}{h}$**.** Figure out how many rows your table needs before you start.
- **Use the slope from the current point, not the next one.** The slope $f(x_n, y_n)$ is evaluated at the *current* position (x_n, y_n), not at (x_{n+1}, y_{n+1}).
- **Carry enough decimal places.** Rounding intermediate values introduces additional error at every step. On the AP exam, keep at least 4 decimal places in intermediate calculations.

- **Know the over/underestimate rule cold.** On AP free-response questions, you are often asked whether your Euler estimate is above or below the true solution. Check whether the solution is concave up (underestimate) or concave down (overestimate) over the interval.

Examples

Example 1: Use Euler's method with $h = 0.5$ to approximate $y(1)$ for $\dfrac{dy}{dx} = x + y$, $y(0) = 1$.

The target is $x = 1$ and $h = 0.5$, so we need $\dfrac{1-0}{0.5} = 2$ steps.

n	x_n	y_n	$f(x_n, y_n) = x_n + y_n$
0	0.0	1.0000	$0.0 + 1.0000 = 1.0000$
1	0.5	$1.0000 + 0.5(1.0000) = 1.5000$	$0.5 + 1.5000 = 2.0000$
2	1.0	$1.5000 + 0.5(2.0000) = 2.5000$	—

$$\boxed{y(1) \approx 2.5000}$$

Note: The exact solution is $y = 2e^x - x - 1$, giving $y(1) = 2e - 2 \approx 3.4366$. The Euler estimate underestimates because the solution is concave up.

Example 2: Use Euler's method with $h = 0.5$ to approximate $y(2)$ for $\dfrac{dy}{dx} = 2x$, $y(0) = 1$.

We need $\dfrac{2-0}{0.5} = 4$ steps.

n	x_n	y_n	$f(x_n, y_n) = 2x_n$
0	0.0	1.0000	$2(0.0) = 0.0000$
1	0.5	$1.0000 + 0.5(0.0000) = 1.0000$	$2(0.5) = 1.0000$
2	1.0	$1.0000 + 0.5(1.0000) = 1.5000$	$2(1.0) = 2.0000$
3	1.5	$1.5000 + 0.5(2.0000) = 2.5000$	$2(1.5) = 3.0000$
4	2.0	$2.5000 + 0.5(3.0000) = 4.0000$	—

$$\boxed{y(2) \approx 4.0000}$$

Note: The exact solution is $y = x^2 + 1$, giving $y(2) = 5$. The Euler estimate underestimates because $y = x^2 + 1$ is concave up.

Example 3: Use Euler's method with $h = 0.25$ to approximate $y(1)$ for $\frac{dy}{dx} = y$, $y(0) = 1$.

We need $\frac{1-0}{0.25} = 4$ steps.

n	x_n	y_n	$f(x_n, y_n) = y_n$
0	0.00	1.0000	1.0000
1	0.25	$1.0000 + 0.25(1.0000) = 1.2500$	1.2500
2	0.50	$1.2500 + 0.25(1.2500) = 1.5625$	1.5625
3	0.75	$1.5625 + 0.25(1.5625) = 1.9531$	1.9531
4	1.00	$1.9531 + 0.25(1.9531) = 2.4414$	—

$$\boxed{y(1) \approx 2.4414}$$

Note: The exact solution is $y = e^x$, so $y(1) = e \approx 2.7183$. With $h = 0.1$ (10 steps), the approximation improves to ≈ 2.5937, confirming that smaller step sizes give better results.

Example 4 (Over/Underestimate): For $\frac{dy}{dx} = y$, $y(0) = 1$, explain whether Euler's method with any $h > 0$ will overestimate or underestimate $y(1)$.

Step 1: The exact solution is $y = e^x$.

Step 2: Check concavity. Differentiate twice:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[y] = y = e^x > 0$$

The solution is **concave up** everywhere.

Step 3: Apply the rule. When the true solution is concave up, the tangent line lies *below* the curve. Since Euler's method follows tangent lines, it will always produce an **underestimate** for this equation.

Common Mistakes

- **Using the slope from the next point instead of the current one:** The formula is $y_{n+1} = y_n + h \cdot f(x_n, y_n)$. The slope is computed at (x_n, y_n) before the step, not at (x_{n+1}, y_{n+1}) after it.

- **Using the wrong number of steps:** Determine the number of steps as $\frac{x_{\text{target}} - x_0}{h}$ before starting. Running too few steps will stop short of the target x-value.

- **Rounding too aggressively in intermediate steps:** Each intermediate y_n feeds into the next step. Rounding to 2 decimal places partway through compounds error. Keep 4 or more decimal places until the final answer.

- **Forgetting to update x at each step:** After computing y_{n+1}, you must also set $x_{n+1} = x_n + h$. When f depends on x, using the wrong x-value in the slope computation gives the wrong slope.
- **Confusing the over/underestimate direction:** Concave up $\Rightarrow$ underestimate. Concave down $\Rightarrow$ overestimate. This rule is for the case where $h > 0$ (stepping forward). Make sure you check the concavity of the *true solution*, not the Euler approximation.

Worksheet

1. Use Euler's method with $h = 0.5$ to approximate $y(1.5)$ for $\dfrac{dy}{dx} = x - y$, $y(0) = 0$.

   Show your work in a table.

2. Use Euler's method with $h = 0.5$ to approximate $y(2)$ for $\dfrac{dy}{dx} = x^2 + 1$, $y(0) = 2$.

   Show your work in a table.

3. Use Euler's method with $h = 0.5$ to approximate $y(2)$ for $\dfrac{dy}{dx} = 2y$, $y(0) = 3$.

   (a) Show your work in a table.

   (b) The exact solution is $y = 3e^{2x}$. Is your Euler estimate an overestimate or underestimate? Explain using concavity.

4. Use Euler's method with $h = 0.5$ to approximate $y(2.5)$ for $\dfrac{dy}{dx} = \dfrac{x}{y}$, $y(1) = 2$.

   Show your work in a table.

5. Use Euler's method with $h = 0.25$ to approximate $y(1)$ for $\dfrac{dy}{dx} = -2y$, $y(0) = 10$.

Show your work in a table.

6. Use Euler's method with $h = 0.25$ to approximate $y(1)$ for $\dfrac{dy}{dx} = x + y$, $y(0) = 0$.

   Show your work in a table.

7. Use Euler's method with $h = 0.5$ to approximate $y(1.5)$ for $\dfrac{dy}{dx} = y - x^2$, $y(0) = 1$.

   Show your work in a table.

8. (Challenge) Use Euler's method with $h = 0.5$ to approximate $y(2)$ for $\frac{dy}{dx} = \sin(y)$, $y(0) = 1$.

   Show your work in a table. Round intermediate values to 4 decimal places.

Answer Key

1. $\dfrac{dy}{dx} = x - y$, $y(0) = 0$, $h = 0.5$, approximate $y(1.5)$.
Number of steps: $\dfrac{1.5 - 0}{0.5} = 3$.

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$
0	0.0	0.0000	$0.0 - 0.0000 = 0.0000$
1	0.5	$0.0000 + 0.5(0.0000) = 0.0000$	$0.5 - 0.0000 = 0.5000$
2	1.0	$0.0000 + 0.5(0.5000) = 0.2500$	$1.0 - 0.2500 = 0.7500$
3	1.5	$0.2500 + 0.5(0.7500) = 0.6250$	—

$$\boxed{y(1.5) \approx 0.6250}$$

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2. $\dfrac{dy}{dx} = x^2 + 1$, $y(0) = 2$, $h = 0.5$, approximate $y(2)$.
Number of steps: $\dfrac{2 - 0}{0.5} = 4$.

n	x_n	y_n	$f(x_n, y_n) = x_n^2 + 1$
0	0.0	2.0000	$(0.0)^2 + 1 = 1.0000$
1	0.5	$2.0000 + 0.5(1.0000) = 2.5000$	$(0.5)^2 + 1 = 1.2500$
2	1.0	$2.5000 + 0.5(1.2500) = 3.1250$	$(1.0)^2 + 1 = 2.0000$
3	1.5	$3.1250 + 0.5(2.0000) = 4.1250$	$(1.5)^2 + 1 = 3.2500$
4	2.0	$4.1250 + 0.5(3.2500) = 5.7500$	—

$$\boxed{y(2) \approx 5.7500}$$

The exact solution is $y = \dfrac{x^3}{3} + x + 2$, giving $y(2) = \dfrac{8}{3} + 2 + 2 = \dfrac{20}{3} \approx 6.6\overline{6}$.

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3. $\dfrac{dy}{dx} = 2y$, $y(0) = 3$, $h = 0.5$, approximate $y(2)$.
Number of steps: $\dfrac{2 - 0}{0.5} = 4$.

(a)

n	x_n	y_n	$f(x_n, y_n) = 2y_n$
0	0.0	3.0000	$2(3.0000) = 6.0000$
1	0.5	$3.0000 + 0.5(6.0000) = 6.0000$	$2(6.0000) = 12.0000$
2	1.0	$6.0000 + 0.5(12.0000) = 12.0000$	$2(12.0000) = 24.0000$
3	1.5	$12.0000 + 0.5(24.0000) = 24.0000$	$2(24.0000) = 48.0000$
4	2.0	$24.0000 + 0.5(48.0000) = 48.0000$	—

$$\boxed{y(2) \approx 48.0000}$$

(b) The exact solution is $y = 3e^{2x}$. Checking concavity:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[2y] = 2\frac{dy}{dx} = 2(2y) = 4y > 0$$

The solution is **concave up**, so Euler's method is an **underestimate**. Indeed, $3e^4 \approx 163.79 \gg 48$.

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4. $\dfrac{dy}{dx} = \dfrac{x}{y}$, $y(1) = 2$, $h = 0.5$, approximate $y(2.5)$.

Number of steps: $\dfrac{2.5 - 1}{0.5} = 3$.

n	x_n	y_n	$f(x_n, y_n) = x_n/y_n$
0	1.0	2.0000	$1.0/2.0000 = 0.5000$
1	1.5	$2.0000 + 0.5(0.5000) = 2.2500$	$1.5/2.2500 = 0.6\overline{6}$
2	2.0	$2.2500 + 0.5(0.6\overline{6}) = 2.5833$	$2.0/2.5833 \approx 0.7742$
3	2.5	$2.5833 + 0.5(0.7742) = 2.9704$	—

$$\boxed{y(2.5) \approx 2.9704}$$

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5. $\dfrac{dy}{dx} = -2y$, $y(0) = 10$, $h = 0.25$, approximate $y(1)$.

Number of steps: $\dfrac{1 - 0}{0.25} = 4$.

n	x_n	y_n	$f(x_n, y_n) = -2y_n$
0	0.00	10.0000	-20.0000
1	0.25	$10.0000 + 0.25(-20.0000) = 5.0000$	-10.0000
2	0.50	$5.0000 + 0.25(-10.0000) = 2.5000$	-5.0000
3	0.75	$2.5000 + 0.25(-5.0000) = 1.2500$	-2.5000
4	1.00	$1.2500 + 0.25(-2.5000) = 0.6250$	—

$$\boxed{y(1) \approx 0.6250}$$

The exact solution is $y = 10e^{-2x}$, giving $y(1) = 10e^{-2} \approx 1.3534$.

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6. $\dfrac{dy}{dx} = x + y$, $y(0) = 0$, $h = 0.25$, approximate $y(1)$.

Number of steps: $\dfrac{1-0}{0.25} = 4$.

n	x_n	y_n	$f(x_n, y_n) = x_n + y_n$
0	0.00	0.0000	0.0000
1	0.25	$0.0000 + 0.25(0.0000) = 0.0000$	$0.25 + 0.0000 = 0.2500$
2	0.50	$0.0000 + 0.25(0.2500) = 0.0625$	$0.50 + 0.0625 = 0.5625$
3	0.75	$0.0625 + 0.25(0.5625) = 0.2031$	$0.75 + 0.2031 = 0.9531$
4	1.00	$0.2031 + 0.25(0.9531) = 0.4414$	—

$$\boxed{y(1) \approx 0.4414}$$

The exact solution is $y = e^x - x - 1$, giving $y(1) = e - 2 \approx 0.7183$.

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7. $\dfrac{dy}{dx} = y - x^2$, $y(0) = 1$, $h = 0.5$, approximate $y(1.5)$.

Number of steps: $\dfrac{1.5-0}{0.5} = 3$.

n	x_n	y_n	$f(x_n, y_n) = y_n - x_n^2$
0	0.0	1.0000	$1.0000 - 0 = 1.0000$
1	0.5	$1.0000 + 0.5(1.0000) = 1.5000$	$1.5000 - 0.25 = 1.2500$
2	1.0	$1.5000 + 0.5(1.2500) = 2.1250$	$2.1250 - 1.00 = 1.1250$
3	1.5	$2.1250 + 0.5(1.1250) = 2.6875$	—

$$\boxed{y(1.5) \approx 2.6875}$$

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8. (Challenge) $\dfrac{dy}{dx} = \sin(y)$, $y(0) = 1$, $h = 0.5$, approximate $y(2)$.

Number of steps: $\dfrac{2-0}{0.5} = 4$.

n	x_n	y_n	$f(x_n, y_n) = \sin(y_n)$
0	0.0	1.0000	$\sin(1.0000) = 0.8415$
1	0.5	$1.0000 + 0.5(0.8415) = 1.4207$	$\sin(1.4207) = 0.9888$
2	1.0	$1.4207 + 0.5(0.9888) = 1.9151$	$\sin(1.9151) = 0.9412$
3	1.5	$1.9151 + 0.5(0.9412) = 2.3857$	$\sin(2.3857) = 0.6858$
4	2.0	$2.3857 + 0.5(0.6858) = 2.7286$	—

$$\boxed{y(2) \approx 2.7286}$$

This DE has no elementary closed-form solution. Euler's method is one of the few practical tools for approximating its solution.

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End of Lesson • ApexMath
