Arc Length
ApexMath

Notes

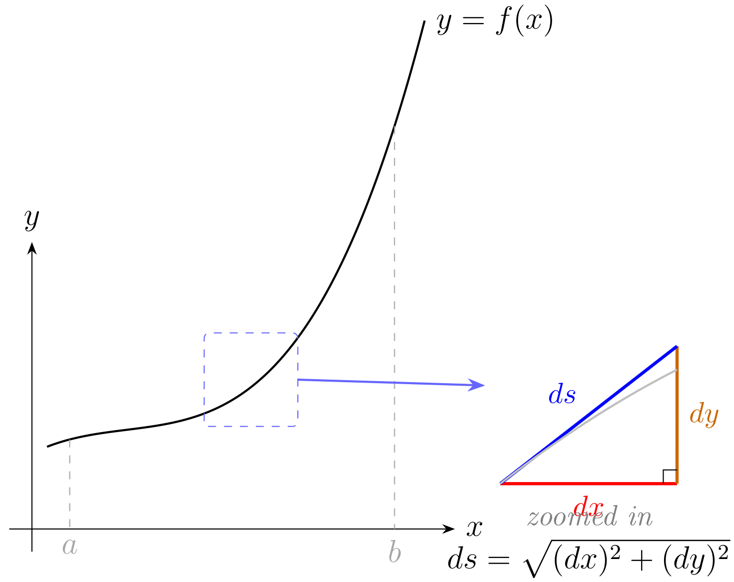
The Big Idea: Measuring the Length of a Curve

So far, you have used integrals to find areas and volumes. Now we use the same idea to measure something different: the *length* of a curved path along a graph.

Imagine you drew the curve $y = f(x)$ on paper and then carefully laid a piece of string along it from $x = a$ to $x = b$. When you pulled the string straight, how long would it be? That is exactly what arc length measures.

Building the Formula from Scratch

The key idea is to zoom in on a tiny piece of the curve. When you zoom in close enough, any smooth curve looks like a straight line segment. We can find the length of that tiny segment using the distance formula, then add up all the tiny pieces with an integral.



Consider a tiny step along the curve. Moving a horizontal distance dx and a vertical distance dy, the tiny segment has length ds given by the Pythagorean theorem:

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

We want to integrate with respect to x, so we factor $(dx)^2$ out from under the square root. Factor $(dx)^2$ out of both terms inside:

$$ds = \sqrt{(dx)^2\left(1 + \frac{(dy)^2}{(dx)^2}\right)} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Since $\frac{dy}{dx} = f'(x)$, we can write:

$$ds = \sqrt{1 + [f'(x)]^2}\, dx$$

Integrating from $x = a$ to $x = b$ adds up all of these tiny lengths to give the total arc length L:

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

What Each Part Means

- $f'(x)$ is the derivative of the curve — it tells you the slope at each point.
- $[f'(x)]^2$ squares that slope.
- $1 + [f'(x)]^2$ adds 1 to the squared slope.
- $\sqrt{1 + [f'(x)]^2}$ is the length of a tiny diagonal segment at that x-value.
- The integral adds up all those tiny lengths from a to b.

The Arc Length Differential

The expression $ds = \sqrt{1 + [f'(x)]^2}\, dx$ is called the *arc length differential.* You will see this same expression appear again in the Surface Area of Revolution lesson, so understanding it here will pay off later.

Pro Tip

- **Find $f'(x)$ first, then square it.** The most common error is forgetting to take the derivative before substituting. Write $f'(x) = \ldots$ on its own line before setting up the integral.
- **You cannot skip simplification.** Before integrating, always expand or simplify $1 + [f'(x)]^2$ as much as possible. Often the expression under the square root simplifies into a perfect square, which removes the square root entirely and makes integration easy.
- **Look for perfect square trinomials.** When $1 + [f'(x)]^2$ simplifies to something like $(ax + b)^2$, the square root disappears. This is intentional in most textbook problems.
- **Not every arc length integral has a nice closed form.** In real applications, arc length integrals often cannot be solved by hand. On assessments, the problem will almost always be designed so that the square root simplifies cleanly.
- **Check your derivative carefully.** A small error in $f'(x)$ affects every step that follows. Always double-check your derivative before squaring it.

Examples

Example 1 (Basic Arc Length): Find the arc length of $f(x) = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$.

Step 1: Find $f'(x)$.

Apply the power rule. Bring down $\frac{3}{2}$ and reduce the exponent:

$$f'(x) = \frac{2}{3} \cdot \frac{3}{2} x^{1/2} = x^{1/2} = \sqrt{x}$$

Step 2: Square $f'(x)$.

$$[f'(x)]^2 = \left(\sqrt{x}\right)^2 = x$$

Step 3: Build the expression under the square root.

$$1 + [f'(x)]^2 = 1 + x$$

Step 4: Set up the integral.

$$L = \int_0^3 \sqrt{1+x}\, dx$$

Step 5: Integrate using the substitution $u = 1 + x$, so $du = dx$.
When $x = 0$: $u = 1$. When $x = 3$: $u = 4$.

$$L = \int_1^4 \sqrt{u}\, du = \int_1^4 u^{1/2}\, du = \left[\frac{2}{3}u^{3/2}\right]_1^4$$

Step 6: Evaluate at the bounds.

$$L = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2} = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{16}{3} - \frac{2}{3}$$

$$\boxed{L = \frac{14}{3}}$$

Example 2 (Perfect Square Simplification): Find the arc length of $f(x) = \frac{x^2}{4} - \frac{\ln x}{2}$ from $x = 1$ to $x = 2$.

Step 1: Find $f'(x)$.

Differentiate term by term. The derivative of $\frac{x^2}{4}$ is $\frac{x}{2}$, and the derivative of $\frac{\ln x}{2}$ is $\frac{1}{2x}$:

$$f'(x) = \frac{x}{2} - \frac{1}{2x}$$

Step 2: Square $f'(x)$.
Use the formula $(A - B)^2 = A^2 - 2AB + B^2$:

$$[f'(x)]^2 = \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = \frac{x^2}{4} - 2 \cdot \frac{x}{2} \cdot \frac{1}{2x} + \frac{1}{4x^2}$$

Simplify the middle term: $2 \cdot \frac{x}{2} \cdot \frac{1}{2x} = \frac{1}{2}$

$$[f'(x)]^2 = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}$$

Step 3: Add 1 and look for a perfect square.

$$1 + [f'(x)]^2 = 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$$

This is a perfect square trinomial of the form $(A+B)^2 = A^2 + 2AB + B^2$:

$$\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

Step 4: Take the square root (the result is positive on $[1, 2]$).

$$\sqrt{1 + [f'(x)]^2} = \frac{x}{2} + \frac{1}{2x}$$

Step 5: Set up and evaluate the integral.

$$L = \int_1^2 \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^2$$

At $x = 2$: $\frac{4}{4} + \frac{\ln 2}{2} = 1 + \frac{\ln 2}{2}$
At $x = 1$: $\frac{1}{4} + \frac{\ln 1}{2} = \frac{1}{4} + 0 = \frac{1}{4}$

$$L = \left(1 + \frac{\ln 2}{2}\right) - \frac{1}{4} = \frac{3}{4} + \frac{\ln 2}{2}$$

$$\boxed{L = \frac{3}{4} + \frac{\ln 2}{2}}$$

Example 3 (Linear Function — Sanity Check): Find the arc length of $f(x) = 3x+1$ from $x = 0$ to $x = 4$.

This example shows that the arc length formula is consistent with what you already know. The graph of $f(x) = 3x + 1$ is a straight line. We can verify our answer using the distance formula at the end.

Step 1: Find $f'(x)$.

The slope of a linear function is its coefficient on x:

$$f'(x) = 3$$

Step 2: Square $f'(x)$.

$$[f'(x)]^2 = 9$$

Step 3: Build the expression under the square root.

$$1 + [f'(x)]^2 = 1 + 9 = 10$$

Since this is a constant, the square root is just $\sqrt{10}$.
Step 4: Set up and evaluate the integral.

$$L = \int_0^4 \sqrt{10}\,dx = \sqrt{10} \cdot x\Big|_0^4 = 4\sqrt{10}$$

$$\boxed{L = 4\sqrt{10}}$$

Sanity check: The endpoints are (0, 1) and (4, 13). The distance between them is $\sqrt{(4-0)^2 + (13-1)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10}$. ✓

Common Mistakes

- **Forgetting to take the derivative:** The formula uses $f'(x)$, not $f(x)$. You must differentiate the function first. Plugging $f(x)$ directly into the formula gives the wrong answer.
- **Not squaring $f'(x)$ before adding 1:** The formula is $1+[f'(x)]^2$, not $1+f'(x)$. The squaring step is essential — it comes directly from the Pythagorean theorem.
- **Trying to simplify $\sqrt{A^2+B^2}$ as $A+B$:** This is *never* correct. $\sqrt{x^2+1} \neq x+1$. You must check whether $1+[f'(x)]^2$ forms a perfect square *as a whole* before taking the root.
- **Forgetting to update bounds in substitution:** When using u-substitution, always convert the limits of integration to the new variable. Forgetting to update the bounds is one of the most common sources of error.
- **Expecting arc length to equal the x-interval width:** The arc length L is always greater than or equal to $b-a$. A flat line ($f'(x) = 0$) gives $L = b-a$, but any curve will be longer.

Worksheet

1. Find the arc length of $f(x) = \dfrac{x^{3/2}}{3}$ from $x = 0$ to $x = 4$.

2. Find the arc length of $f(x) = \dfrac{x^2}{2}$ from $x = 0$ to $x = 2$.

3. Find the arc length of $f(x) = 2x - 5$ from $x = 1$ to $x = 4$.

4. Find the arc length of $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ from $x = 1$ to $x = 3$.

5. Find the arc length of $f(x) = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$.

6. Find the arc length of $f(x) = \frac{x^4}{8} + \frac{1}{4x^2}$ from $x = 1$ to $x = 2$.

7. (Challenge) Set up, but do not evaluate, the arc length integral for $f(x) = \sin x$ from $x = 0$ to $x = \pi$. Explain in words why this integral cannot be simplified using algebra.

8. (Challenge) Find the arc length of $f(x) = \dfrac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$.

Answer Key

1. $f(x) = \dfrac{x^{3/2}}{3}$ from $x = 0$ to $x = 4$.

Step 1: Find $f'(x)$.

Apply the power rule: bring down $\dfrac{3}{2}$ and reduce the exponent by 1:

$$f'(x) = \frac{1}{3} \cdot \frac{3}{2}\, x^{1/2} = \frac{1}{2}x^{1/2} = \frac{\sqrt{x}}{2}$$

Step 2: Square $f'(x)$.

$$[f'(x)]^2 = \left(\frac{\sqrt{x}}{2}\right)^2 = \frac{x}{4}$$

Step 3: Build the expression under the square root.

$$1 + [f'(x)]^2 = 1 + \frac{x}{4} = \frac{4+x}{4}$$

Step 4: Set up the integral.

$$L = \int_0^4 \sqrt{\frac{4+x}{4}}\, dx = \int_0^4 \frac{\sqrt{4+x}}{2}\, dx = \frac{1}{2}\int_0^4 \sqrt{4+x}\, dx$$

Step 5: Substitute $u = 4 + x$, $du = dx$.
When $x = 0$: $u = 4$. When $x = 4$: $u = 8$.

$$L = \frac{1}{2}\int_4^8 \sqrt{u}\, du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^8 = \frac{1}{3}\left[u^{3/2}\right]_4^8$$

Step 6: Evaluate at the bounds.

$$L = \frac{1}{3}\left[(8)^{3/2} - (4)^{3/2}\right] = \frac{1}{3}\left[16\sqrt{2} - 8\right]$$

$$\boxed{L = \frac{16\sqrt{2} - 8}{3}}$$

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2. $f(x) = \dfrac{x^2}{2}$ from $x = 0$ to $x = 2$.

Step 1: Find $f'(x)$.

$$f'(x) = x$$

Step 2: Square $f'(x)$.

$$[f'(x)]^2 = x^2$$

Step 3: Build the expression under the square root.

$$1 + [f'(x)]^2 = 1 + x^2$$

Step 4: Set up the integral.

$$L = \int_0^2 \sqrt{1+x^2}\,dx$$

This integral does not simplify to a perfect square. The exact closed form uses the formula $\displaystyle\int \sqrt{1+x^2}\,dx = \frac{x\sqrt{1+x^2}}{2} + \frac{\ln(x+\sqrt{1+x^2})}{2} + C$.
Evaluating at the bounds:

$$L = \left[\frac{x\sqrt{1+x^2}}{2} + \frac{\ln\left(x+\sqrt{1+x^2}\right)}{2}\right]_0^2$$

At $x = 2$: $\dfrac{2\sqrt{5}}{2} + \dfrac{\ln(2+\sqrt{5})}{2} = \sqrt{5} + \dfrac{\ln(2+\sqrt{5})}{2}$
At $x = 0$: $0 + \dfrac{\ln 1}{2} = 0$

$$\boxed{L = \sqrt{5} + \frac{\ln(2+\sqrt{5})}{2} \approx 2.958}$$

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3. $f(x) = 2x - 5$ from $x = 1$ to $x = 4$.

Step 1: Find $f'(x)$.
The function is linear. Its derivative is just its slope:

$$f'(x) = 2$$

Step 2: Square $f'(x)$.

$$[f'(x)]^2 = 4$$

Step 3: Build the expression under the square root.

$$1 + [f'(x)]^2 = 1 + 4 = 5$$

This is a constant, so $\sqrt{5}$ comes out of the integral.
Step 4: Integrate.

$$L = \int_1^4 \sqrt{5}\,dx = \sqrt{5}\cdot[x]_1^4 = \sqrt{5}(4-1) = 3\sqrt{5}$$

$$\boxed{L = 3\sqrt{5}}$$

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4. $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ from $x = 1$ to $x = 3$.

Step 1: Find $f'(x)$.

Differentiate each term. For $\frac{1}{2x} = \frac{1}{2}x^{-1}$, the derivative is $-\frac{1}{2}x^{-2}$:

$$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$$

Step 2: Square $f'(x)$.
Use $(A-B)^2 = A^2 - 2AB + B^2$:

$$[f'(x)]^2 = \frac{x^4}{4} - 2 \cdot \frac{x^2}{2} \cdot \frac{1}{2x^2} + \frac{1}{4x^4} = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$$

Step 3: Add 1 and look for a perfect square.

$$1 + [f'(x)]^2 = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

Step 4: Take the square root.

$$\sqrt{1 + [f'(x)]^2} = \frac{x^2}{2} + \frac{1}{2x^2}$$

Step 5: Integrate.

$$L = \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^3$$

At $x = 3$: $\frac{27}{6} - \frac{1}{6} = \frac{26}{6} = \frac{13}{3}$
At $x = 1$: $\frac{1}{6} - \frac{3}{6} = -\frac{2}{6} = -\frac{1}{3}$

$$L = \frac{13}{3} - \left(-\frac{1}{3}\right) = \frac{14}{3}$$

$$\boxed{L = \frac{14}{3}}$$

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5. $f(x) = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$.

Step 1: Find $f'(x)$.

Use the chain rule. The derivative of $\ln(u)$ is $\frac{1}{u} \cdot u'$:

$$f'(x) = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$$

Step 2: Square $f'(x)$.

$$[f'(x)]^2 = \tan^2 x$$

Step 3: Build the expression under the square root.
Use the Pythagorean identity $1 + \tan^2 x = \sec^2 x$:

$$1 + [f'(x)]^2 = 1 + \tan^2 x = \sec^2 x$$

Step 4: Take the square root.
Since $\sec x > 0$ on $\left[0, \frac{\pi}{4}\right]$:

$$\sqrt{\sec^2 x} = \sec x$$

Step 5: Integrate.

$$L = \int_0^{\pi/4} \sec x \, dx = \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

At $x = \frac{\pi}{4}$: $\sec\frac{\pi}{4} = \sqrt{2}$, $\tan\frac{\pi}{4} = 1$, so the value is $\ln(\sqrt{2}+1)$.
At $x = 0$: $\sec 0 = 1$, $\tan 0 = 0$, so the value is $\ln(1) = 0$.

$$L = \ln(\sqrt{2}+1) - 0$$

$$\boxed{L = \ln(\sqrt{2}+1)}$$

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6. $f(x) = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ from $x = 1$ to $x = 2$.

Step 1: Find $f'(x)$.
Write $\dfrac{1}{4x^2} = \dfrac{1}{4}x^{-2}$, so its derivative is $-\dfrac{1}{2}x^{-3} = -\dfrac{1}{2x^3}$:

$$f'(x) = \frac{x^3}{2} - \frac{1}{2x^3}$$

Step 2: Square $f'(x)$.
Use $(A-B)^2 = A^2 - 2AB + B^2$:

$$[f'(x)]^2 = \frac{x^6}{4} - 2 \cdot \frac{x^3}{2} \cdot \frac{1}{2x^3} + \frac{1}{4x^6} = \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}$$

Step 3: Add 1 and identify the perfect square.

$$1 + [f'(x)]^2 = \frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6} = \left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2$$

Step 4: Take the square root.

$$\sqrt{1 + [f'(x)]^2} = \frac{x^3}{2} + \frac{1}{2x^3}$$

Step 5: Integrate.

$$L = \int_1^2 \left(\frac{x^3}{2} + \frac{1}{2x^3}\right) dx = \left[\frac{x^4}{8} - \frac{1}{4x^2}\right]_1^2$$

At $x = 2$: $\dfrac{16}{8} - \dfrac{1}{16} = 2 - \dfrac{1}{16} = \dfrac{31}{16}$

At $x = 1$: $\dfrac{1}{8} - \dfrac{1}{4} = -\dfrac{1}{8}$

$$L = \frac{31}{16} - \left(-\frac{1}{8}\right) = \frac{31}{16} + \frac{2}{16} = \frac{33}{16}$$

$$\boxed{L = \frac{33}{16}}$$

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7. (Challenge) Set up the arc length integral for $f(x) = \sin x$ from $x = 0$ to $x = \pi$.

Step 1: Find $f'(x)$.

$$f'(x) = \cos x$$

Step 2: Square $f'(x)$.

$$[f'(x)]^2 = \cos^2 x$$

Step 3: Build the expression under the square root.

$$1 + [f'(x)]^2 = 1 + \cos^2 x$$

Step 4: Set up the integral.

$$L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx$$

This integral cannot be simplified. The expression $\sqrt{1 + \cos^2 x}$ is not a perfect square and has no elementary antiderivative. It must be evaluated numerically ($L \approx 3.820$). This is an example of an arc length integral that arises naturally but cannot be solved by hand.

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8. (Challenge) $f(x) = \dfrac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$.

Step 1: Find $f'(x)$ using the chain rule.

Let $u = x^2 + 2$, so $f(x) = \dfrac{1}{3}u^{3/2}$.

By the chain rule:

$$f'(x) = \frac{1}{3} \cdot \frac{3}{2}(x^2 + 2)^{1/2} \cdot 2x = x(x^2 + 2)^{1/2} = x\sqrt{x^2 + 2}$$

Step 2: Square $f'(x)$.

$$[f'(x)]^2 = x^2(x^2 + 2) = x^4 + 2x^2$$

Step 3: Build the expression under the square root.

$$1 + [f'(x)]^2 = 1 + x^4 + 2x^2 = x^4 + 2x^2 + 1 = (x^2 + 1)^2$$

This is a perfect square.
Step 4: Take the square root.

$$\sqrt{(x^2+1)^2} = x^2 + 1$$

Step 5: Integrate.

$$L = \int_0^3 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_0^3 = \frac{27}{3} + 3 - 0 = 9 + 3$$

$$\boxed{L = 12}$$

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End of Lesson • ApexMath
