Area Between Curves
ApexMath

Notes

The Big Idea: Building Area from a Thin Strip

Instead of memorizing a formula, we build it from scratch. Imagine slicing the region between two curves into infinitely many thin vertical strips. Each strip has:

- Width: dx (infinitely thin)
- Height: the distance between the top curve and the bottom curve at that x-value

The area of one thin strip is:

$$dA = [\text{top} - \text{bottom}]\, dx = [f(x) - g(x)]\, dx$$

To get the total area, we add up (integrate) all the strips:

$$A = \int dA = \int_a^b [f(x) - g(x)]\, dx$$

This is the foundation of every area-between-curves problem. Always start by identifying the strip, writing dA, then integrating.

The Key Rule: Top Minus Bottom

The height of the strip is always **top minus bottom**. This guarantees the area is positive, as long as you correctly identify which function is on top. If $f(x) \geq g(x)$ on $[a, b]$, then:

$$dA = [f(x) - g(x)]\, dx \quad \Longrightarrow \quad A = \int_a^b [f(x) - g(x)]\, dx$$

If you put the bottom function minus the top, the height becomes negative and you get a negative area. Always double-check which curve is higher by testing a point.

Finding the Limits: Intersection Points

When limits are not given, set the two functions equal to each other and solve:

$$f(x) = g(x)$$

The solutions are the x-values where the curves cross — these become your limits of integration a and b.

When the Curves Switch Positions

If the curves cross each other inside the interval, the top and bottom functions swap at the crossing point. You cannot use a single dA expression for the whole interval. Instead:

1. Find all crossing points $x = c$ inside $[a, b]$.
2. Write a separate dA for each subinterval with the correct top minus bottom.
3. Integrate each piece and add.

Horizontal Strips: Integrating with Respect to y

Sometimes it is easier to slice the region into **horizontal** strips instead of vertical ones. Each horizontal strip has:

- Height: dy (infinitely thin)
- Width: the distance between the right curve and the left curve at that y-value

The area of one thin horizontal strip is:

$$dA = [\text{right} - \text{left}]\, dy = [R(y) - L(y)]\, dy$$

Integrating over the y-range of the region:

$$A = \int dA = \int_c^d [R(y) - L(y)]\, dy$$

Use horizontal strips when the boundary curves are naturally expressed as $x =$ (a function of y), or when vertical strips would require splitting the region into multiple pieces.

Pro Tip

- Always write dA explicitly before integrating. Writing the strip out forces you to identify top, bottom, and direction before setting up the integral.
- After finding intersection points, test a value between them to confirm which function is on top. Never assume — always verify with a quick substitution.
- If a region is described by two functions and you need to decide between vertical or horizontal strips, ask: which approach gives one clean dA expression without needing to split? That is the better choice.
- For horizontal strips, you may need to rewrite $y = f(x)$ as $x = f(y)$ first. Solve for x explicitly before writing dA.
- When curves switch positions, the area of each piece is positive. If one of your integrals comes out negative after splitting, you have the top and bottom reversed in that piece.

Examples

Example 1 (Given Limits): Find the area between $f(x) = x^2 + 1$ and $g(x) = x - 1$ on $[0, 2]$.

Step 1: Identify top and bottom. Test $x = 1$:

$$f(1) = 2, \quad g(1) = 0$$

Since $f(1) > g(1)$, f is on top.
Step 2: Write dA.

$$dA = [f(x) - g(x)]\,dx = [(x^2 + 1) - (x - 1)]\,dx = (x^2 - x + 2)\,dx$$

Step 3: Integrate.

$$A = \int_0^2 (x^2 - x + 2)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_0^2$$

At $x = 2$: $\dfrac{8}{3} - 2 + 4 = \dfrac{8}{3} + 2 = \dfrac{14}{3}$
At $x = 0$: 0

$$\boxed{A = \frac{14}{3}}$$

Example 2 (Find Intersection Points First): Find the area enclosed between $f(x) = 4 - x^2$ and $g(x) = x + 2$.

Step 1: Find intersection points.

$$4 - x^2 = x + 2 \implies x^2 + x - 2 = 0 \implies (x + 2)(x - 1) = 0$$

$$x = -2 \quad \text{and} \quad x = 1$$

Step 2: Identify top and bottom. Test $x = 0$:

$$f(0) = 4, \quad g(0) = 2$$

f is on top on $[-2, 1]$.
Step 3: Write dA.

$$dA = [f(x) - g(x)]\,dx = [(4 - x^2) - (x + 2)]\,dx = (2 - x - x^2)\,dx$$

Step 4: Integrate.

$$A = \int_{-2}^{1} (2 - x - x^2)\,dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

At $x = 1$: $2 - \frac{1}{2} - \frac{1}{3} = \frac{12 - 3 - 2}{6} = \frac{7}{6}$
At $x = -2$: $-4 - 2 + \frac{8}{3} = -6 + \frac{8}{3} = \frac{-18 + 8}{3} = -\frac{10}{3}$

$$A = \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{7}{6} + \frac{20}{6} = \frac{27}{6}$$

$$\boxed{A = \frac{9}{2}}$$

Example 3 (Curves Switch Positions): Find the area enclosed between $f(x) = x^3$ and $g(x) = x$ on $[-1, 1]$.

Step 1: Find intersection points.

$$x^3 = x \implies x^3 - x = 0 \implies x(x-1)(x+1) = 0$$

$$x = -1, \quad x = 0, \quad x = 1$$

Step 2: Determine which is on top on each subinterval.
Test $x = -0.5$: $f(-0.5) = -0.125$, $g(-0.5) = -0.5$. So $f > g$ on $[-1, 0]$.
Test $x = 0.5$: $f(0.5) = 0.125$, $g(0.5) = 0.5$. So $g > f$ on $[0, 1]$.
Step 3: Write a separate dA for each piece.
On $[-1, 0]$: $dA_1 = [x^3 - x]\,dx$
On $[0, 1]$: $dA_2 = [x - x^3]\,dx$
Step 4: Integrate each piece.

$$\int_{-1}^{0} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$$

$$\int_{0}^{1} (x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Step 5: Add the pieces.

$$A = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\boxed{A = \frac{1}{2}}$$

Example 4 (Horizontal Strips): Find the area enclosed between $x = y^2$ and $x = 2 - y^2$.

Step 1: Find intersection points in y.

$$y^2 = 2 - y^2 \implies 2y^2 = 2 \implies y^2 = 1 \implies y = \pm 1$$

Step 2: Identify right and left. Test $y = 0$:

$$R(0) = 2 - 0 = 2, \quad L(0) = 0$$

$x = 2 - y^2$ is the right boundary, $x = y^2$ is the left boundary.

Step 3: Write dA using horizontal strips.

$$dA = [R(y) - L(y)]\,dy = [(2 - y^2) - y^2]\,dy = (2 - 2y^2)\,dy$$

Step 4: Integrate from $y = -1$ to $y = 1$.

$$A = \int_{-1}^{1} (2 - 2y^2)\,dy = \left[2y - \frac{2y^3}{3}\right]_{-1}^{1}$$

At $y = 1$: $2 - \dfrac{2}{3} = \dfrac{4}{3}$

At $y = -1$: $-2 + \dfrac{2}{3} = -\dfrac{4}{3}$

$$A = \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}$$

$$\boxed{A = \frac{8}{3}}$$

Example 5 (Three Curves): Find the area of the region bounded by $y = x^2$, $y = 4$, and $y = 0$ (above the x-axis and below $y = 4$).

Step 1: Sketch the region. The parabola $y = x^2$ meets $y = 4$ at $x = \pm 2$. Use horizontal strips since the left and right boundaries are symmetric.

Step 2: Express the boundaries as functions of y.

From $y = x^2$: $x = \pm\sqrt{y}$. The right boundary is $x = \sqrt{y}$ and the left boundary is $x = -\sqrt{y}$.

Step 3: Write dA.

$$dA = [\sqrt{y} - (-\sqrt{y})]\,dy = 2\sqrt{y}\,dy$$

Step 4: Integrate from $y = 0$ to $y = 4$.

$$A = \int_0^4 2\sqrt{y}\,dy = \int_0^4 2y^{1/2}\,dy = \left[\frac{2y^{3/2}}{3/2}\right]_0^4 = \left[\frac{4}{3}y^{3/2}\right]_0^4$$

$$= \frac{4}{3}(4)^{3/2} - 0 = \frac{4}{3}(8) = \frac{32}{3}$$

$$\boxed{A = \frac{32}{3}}$$

Common Mistakes

- **Getting top and bottom backwards:** If the top function is subtracted from the bottom, the area comes out negative. Always test a point between the limits to confirm which function is higher before writing dA.
- **Using a single integral when curves cross:** If the two curves intersect inside the interval, one function is on top for part of the interval and on the bottom for the rest. Using a single dA expression across the full interval mixes the signs and gives the wrong area.
- **Forgetting to find all intersection points:** Set $f(x) = g(x)$ and solve completely. If there are two intersection points, both are needed — one for each limit. Partial solutions lead to wrong limits.
- **Using vertical strips when horizontal strips are cleaner:** If the region has a left boundary and a right boundary that are naturally expressed as $x = h(y)$, forcing vertical strips often requires splitting the integral unnecessarily. Recognize when $dA = [R(y) - L(y)]\,dy$ is the better choice.
- **Forgetting to rewrite $y = f(x)$ as $x = f(y)$:** For horizontal strips, the boundaries must be expressed as functions of y. Solve for x explicitly before writing dA.

Worksheet

1. Find the area between $f(x) = x^2$ and $g(x) = x$ on $[0, 1]$.

2. Find the area enclosed between $f(x) = 6 - x^2$ and $g(x) = x$.

   (Find intersection points first.)

3. Find the area enclosed between $f(x) = x^2 - 4x$ and $g(x) = -x^2 + 4x - 4$.

4. Find the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on $[0, \pi/2]$.

5. Find the total area enclosed between $f(x) = x^3 - x$ and $g(x) = 0$ (the x-axis) on $[-1, 1]$.

   (The curves switch — split and use separate dA expressions.)

6. Find the area of the region bounded by $x = y^2 - 1$ and $x = 3$.

   (Use horizontal strips.)

7. Find the area enclosed between $f(x) = e^x$, $g(x) = e^{-x}$, and $x = 1$.
   (Curves intersect at $x = 0$. Use vertical strips from 0 to 1.)

8. Find the area of the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$.

9. (Challenge) Find the area enclosed between $f(x) = x^2 - 2x$ and $g(x) = x + 4$.

10. (Challenge) Find the area of the region bounded by $x = y^2 - 2y$ and $x = y + 4$ using horizontal strips.

Answer Key

1. $f(x) = x^2$, $g(x) = x$ on $[0, 1]$.
Test $x = 0.5$: $f = 0.25$, $g = 0.5$. So g is on top.
$dA = [x - x^2]\,dx$

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\boxed{A = \frac{1}{6}}$$

2. $f(x) = 6 - x^2$, $g(x) = x$.
Intersections: $6 - x^2 = x \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0 \Rightarrow x = -3, 2$.
Test $x = 0$: $f = 6$, $g = 0$. f is on top.
$dA = [(6 - x^2) - x]\,dx = (6 - x - x^2)\,dx$

$$A = \int_{-3}^{2} (6 - x - x^2)\,dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

At $x = 2$: $12 - 2 - \frac{8}{3} = 10 - \frac{8}{3} = \frac{22}{3}$
At $x = -3$: $-18 - \frac{9}{2} + 9 = -9 - \frac{9}{2} = -\frac{27}{2}$

$$A = \frac{22}{3} - \left(-\frac{27}{2}\right) = \frac{44}{6} + \frac{81}{6} = \frac{125}{6}$$

$$\boxed{A = \frac{125}{6}}$$

3. $f(x) = x^2 - 4x$, $g(x) = -x^2 + 4x - 4$.
Intersections: $x^2 - 4x = -x^2 + 4x - 4 \Rightarrow 2x^2 - 8x + 4 = 0 \Rightarrow x^2 - 4x + 2 = 0$

$$x = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

Test $x = 2$: $f(2) = 4 - 8 = -4$, $g(2) = -4 + 8 - 4 = 0$. So g is on top.
$dA = [g(x) - f(x)]\,dx = [(-x^2 + 4x - 4) - (x^2 - 4x)]\,dx = (-2x^2 + 8x - 4)\,dx$

$$A = \int_{2-\sqrt{2}}^{2+\sqrt{2}} (-2x^2 + 8x - 4)\,dx = \left[-\frac{2x^3}{3} + 4x^2 - 4x\right]_{2-\sqrt{2}}^{2+\sqrt{2}}$$

Let $u = x - 2$, so limits become $-\sqrt{2}$ to $\sqrt{2}$ and the integrand becomes:

$$-2(u+2)^2 + 8(u+2) - 4 + \ldots$$

It is cleaner to evaluate directly. Using the substitution result:

$$A = \frac{8\sqrt{2}}{3} \cdot 2 - \ldots$$

Direct computation:
At $x = 2 + \sqrt{2}$:

$$-\frac{2(2+\sqrt{2})^3}{3} + 4(2+\sqrt{2})^2 - 4(2+\sqrt{2})$$

$$(2+\sqrt{2})^2 = 6 + 4\sqrt{2},\ (2+\sqrt{2})^3 = (2+\sqrt{2})(6+4\sqrt{2}) = 12 + 8\sqrt{2} + 6\sqrt{2} + 8 = 20 + 14\sqrt{2}$$

$$= -\frac{2(20+14\sqrt{2})}{3} + 4(6+4\sqrt{2}) - 4(2+\sqrt{2}) = -\frac{40+28\sqrt{2}}{3} + 24 + 16\sqrt{2} - 8 - 4\sqrt{2}$$

$$= -\frac{40+28\sqrt{2}}{3} + 16 + 12\sqrt{2} = \frac{-40 - 28\sqrt{2} + 48 + 36\sqrt{2}}{3} = \frac{8+8\sqrt{2}}{3}$$

At $x = 2 - \sqrt{2}$: by symmetry (the integrand is symmetric about $x = 2$), the value is $\frac{8-8\sqrt{2}}{3}$.

$$A = \frac{8+8\sqrt{2}}{3} - \frac{8-8\sqrt{2}}{3} = \frac{16\sqrt{2}}{3}$$

$$\boxed{A = \frac{16\sqrt{2}}{3}}$$

4. $f(x) = \sin(x)$, $g(x) = \cos(x)$ on $[0, \pi/2]$.
Test $x = \pi/4$: both equal $\frac{\sqrt{2}}{2}$ — they intersect there. They cross at $x = \pi/4$.
On $[0, \pi/4]$: test $x = 0$: $\sin(0) = 0$, $\cos(0) = 1$. cos is on top.
$dA_1 = [\cos(x) - \sin(x)]\,dx$
On $[\pi/4, \pi/2]$: test $x = \pi/2$: $\sin(\pi/2) = 1 > \cos(\pi/2) = 0$. sin is on top.
$dA_2 = [\sin(x) - \cos(x)]\,dx$

$$\int_0^{\pi/4} (\cos x - \sin x)\,dx = \Big[\sin x + \cos x\Big]_0^{\pi/4} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1) = \sqrt{2} - 1$$

$$\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} = (0-1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = -1 + \sqrt{2}$$

$$A = (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2\sqrt{2} - 2$$

$$\boxed{A = 2\sqrt{2} - 2}$$

5. $f(x) = x^3 - x$, $g(x) = 0$ on $[-1, 1]$. Zeros at $x = -1, 0, 1$.

On $[-1,0]$: test $x=-0.5$: $f(-0.5)=0.375>0$. f on top.
$dA_1=[x^3-x]\,dx$
On $[0,1]$: test $x=0.5$: $f(0.5)=-0.375<0$. $g=0$ on top.
$dA_2=[0-(x^3-x)]\,dx=[x-x^3]\,dx$

$$\int_{-1}^{0}(x^3-x)\,dx=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}=0-\left(\frac{1}{4}-\frac{1}{2}\right)=\frac{1}{4}$$

$$\int_{0}^{1}(x-x^3)\,dx=\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1}=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

$$A=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\boxed{A=\frac{1}{2}}$$

6. $x=y^2-1$ and $x=3$ using horizontal strips.
Intersections: $y^2-1=3\Rightarrow y^2=4\Rightarrow y=\pm 2$.
Right boundary: $x=3$. Left boundary: $x=y^2-1$.
$dA=[3-(y^2-1)]\,dy=[4-y^2]\,dy$

$$A=\int_{-2}^{2}(4-y^2)\,dy=\left[4y-\frac{y^3}{3}\right]_{-2}^{2}$$

At $y=2$: $8-\frac{8}{3}=\frac{16}{3}$. At $y=-2$: $-8+\frac{8}{3}=-\frac{16}{3}$.

$$A=\frac{16}{3}+\frac{16}{3}=\frac{32}{3}$$

$$\boxed{A=\frac{32}{3}}$$

7. $f(x)=e^x$, $g(x)=e^{-x}$ from $x=0$ to $x=1$.
Test $x=0$: $f=1=g$. They meet at $x=0$. Test $x=0.5$: $f=e^{0.5}>e^{-0.5}=g$. f is on top on $[0,1]$.
$dA=[e^x-e^{-x}]\,dx$

$$A=\int_{0}^{1}(e^x-e^{-x})\,dx=\left[e^x+e^{-x}\right]_{0}^{1}=(e+e^{-1})-(1+1)=e+\frac{1}{e}-2$$

$$\boxed{A=e+\frac{1}{e}-2}$$

8. $y=\sqrt{x}$, $y=0$, $x=4$.
f is on top ($\sqrt{x}\geq 0$). $dA=[\sqrt{x}-0]\,dx=\sqrt{x}\,dx$

$$A = \int_0^4 \sqrt{x}\, dx = \left[\frac{2x^{3/2}}{3}\right]_0^4 = \frac{2(8)}{3} = \frac{16}{3}$$

$$\boxed{A = \frac{16}{3}}$$

9. (Challenge) $f(x) = x^2 - 2x$, $g(x) = x + 4$.
Intersections: $x^2 - 2x = x + 4 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0 \Rightarrow x = -1, 4$.
Test $x = 0$: $f(0) = 0$, $g(0) = 4$. g is on top.
$dA = [(x+4) - (x^2 - 2x)]\, dx = (-x^2 + 3x + 4)\, dx$

$$A = \int_{-1}^4 (-x^2 + 3x + 4)\, dx = \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x\right]_{-1}^4$$

At $x = 4$: $-\frac{64}{3} + 24 + 16 = 40 - \frac{64}{3} = \frac{120 - 64}{3} = \frac{56}{3}$
At $x = -1$: $\frac{1}{3} + \frac{3}{2} - 4 = \frac{1}{3} + \frac{3}{2} - 4 = \frac{2 + 9 - 24}{6} = -\frac{13}{6}$

$$A = \frac{56}{3} - \left(-\frac{13}{6}\right) = \frac{112}{6} + \frac{13}{6} = \frac{125}{6}$$

$$\boxed{A = \frac{125}{6}}$$

10. (Challenge) $x = y^2 - 2y$, $x = y + 4$ using horizontal strips.
Intersections: $y^2 - 2y = y + 4 \Rightarrow y^2 - 3y - 4 = 0 \Rightarrow (y-4)(y+1) = 0 \Rightarrow y = -1, 4$.
Test $y = 0$: $L = y^2 - 2y = 0$, $R = y + 4 = 4$. Right boundary is $x = y + 4$.
$dA = [(y+4) - (y^2 - 2y)]\, dy = (-y^2 + 3y + 4)\, dy$

$$A = \int_{-1}^4 (-y^2 + 3y + 4)\, dy = \left[-\frac{y^3}{3} + \frac{3y^2}{2} + 4y\right]_{-1}^4$$

At $y = 4$: $-\frac{64}{3} + 24 + 16 = \frac{56}{3}$
At $y = -1$: $\frac{1}{3} + \frac{3}{2} - 4 = -\frac{13}{6}$

$$A = \frac{56}{3} + \frac{13}{6} = \frac{112 + 13}{6} = \frac{125}{6}$$

$$\boxed{A = \frac{125}{6}}$$

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