Area Under a Curve
ApexMath

Notes

What Does the Definite Integral Measure?

The definite integral $\displaystyle\int_a^b f(x)\,dx$ measures the **signed area** between the curve $y = f(x)$ and the x-axis over the interval $[a, b]$.

- When $f(x) > 0$, the curve is **above** the x-axis and the integral contributes **positive** area.
- When $f(x) < 0$, the curve is **below** the x-axis and the integral contributes **negative** area.

This distinction is critical. If you blindly evaluate $\int_a^b f(x)\,dx$ when f dips below the x-axis, positive and negative regions cancel each other out, giving you less than the true geometric area.

Signed Area vs. Total Area
Signed area (what the definite integral gives you directly):

$$\text{Signed Area} = \int_a^b f(x)\,dx$$

Regions above the x-axis add to the total; regions below subtract from it.

Total area (the actual geometric area, always non-negative):

$$\text{Total Area} = \int_a^b |f(x)|\,dx$$

To evaluate this, you must:

1. Find all zeros of $f(x)$ on $[a, b]$ — these are the points where the curve crosses the x-axis.
2. Split the integral at each zero.
3. Take the absolute value of each piece (or flip the sign of pieces where $f(x) < 0$).
4. Add all the pieces together.

The Process in Detail
Suppose $f(x)$ has a zero at $x = c$ inside $[a, b]$, and $f(x) > 0$ on $[a, c]$ while $f(x) < 0$ on $[c, b]$.

Then:

$$\text{Total Area} = \int_a^c f(x)\,dx + \int_c^b |f(x)|\,dx = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

The key is that any piece where f is negative must be **negated** (multiplied by -1) to convert it from signed area to positive geometric area.

Setting Up the Integral from a Description
Sometimes you are not given explicit limits and must find them from context:

- **Bounded by the x-axis**: the limits are the x-intercepts of $f(x)$. Set $f(x) = 0$ and solve.
- **Between two specific x-values**: the limits are given directly.
- **Enclosed region**: find all intersection points with the x-axis and use those as your limits.

Pro Tip

- Before evaluating any area integral, sketch the function or test a point in each subinterval to confirm whether $f(x)$ is positive or negative there. This tells you whether each piece needs to be flipped.
- When a problem says "find the area of the region bounded by $f(x)$ and the x-axis," it always means total (positive) area. You must split at zeros.
- When a problem says "evaluate $\int_a^b f(x)\,dx$," it is asking for signed area. No splitting needed — just integrate and report the value, even if it is negative.
- A quick sign check: after integrating a piece where you know $f(x) < 0$, the result should be negative. If it comes out positive, you made a sign error.
- Always double-check that all zeros inside $[a, b]$ have been found and used as split points. Missing a crossing means one of your "positive" pieces actually contains a negative region, giving the wrong total.

Examples

Example 1 (Simple Area Above the x-axis): Find the area under $f(x) = 4 - x^2$ from $x = -2$ to $x = 2$.

Step 1: Check the sign of $f(x)$ on $[-2, 2]$.
Test $x = 0$: $f(0) = 4 > 0$. Check the zeros: $4 - x^2 = 0 \Rightarrow x = \pm 2$. So $f(x) \geq 0$ on all of $[-2, 2]$ — no splitting needed.

Step 2: Set up and evaluate the integral.

$$\int_{-2}^{2} (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

At $x = 2$: $8 - \frac{8}{3} = \frac{24 - 8}{3} = \frac{16}{3}$
At $x = -2$: $-8 + \frac{8}{3} = \frac{-24 + 8}{3} = -\frac{16}{3}$

$$\frac{16}{3} - \left(-\frac{16}{3}\right) = \frac{32}{3}$$

$$\boxed{\text{Area} = \frac{32}{3}}$$

Example 2 (Signed Area vs. Total Area): For $f(x) = x^3 - x$ on $[-1, 1]$:

(a) Find the signed area $\int_{-1}^{1} (x^3 - x)\, dx$.

(b) Find the total area enclosed between the curve and the x-axis.

Part (a): Signed Area.

$$\int_{-1}^{1} (x^3 - x)\, dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1}$$

At $x = 1$: $\frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$
At $x = -1$: $\frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$$-\frac{1}{4} - \left(-\frac{1}{4}\right) = 0$$

$$\boxed{\text{Signed Area} = 0}$$

Note: $f(x) = x^3 - x$ is an odd function, so positive and negative regions cancel exactly over a symmetric interval.

Part (b): Total Area.
Find the zeros of $f(x) = x^3 - x = x(x - 1)(x + 1)$: zeros at $x = -1, 0, 1$.
Test signs:

- On $[-1, 0]$: test $x = -0.5$: $f(-0.5) = (-0.5)((-0.5)^2 - 1) = (-0.5)(-0.75) = 0.375 > 0$
- On $[0, 1]$: test $x = 0.5$: $f(0.5) = (0.5)(0.25 - 1) = (0.5)(-0.75) = -0.375 < 0$

Split at $x = 0$:

$$\text{Total Area} = \int_{-1}^{0} (x^3 - x)\,dx - \int_{0}^{1} (x^3 - x)\,dx$$

$$\int_{-1}^{0} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$$

$$\int_{0}^{1} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

$$\text{Total Area} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\boxed{\text{Total Area} = \frac{1}{2}}$$

Example 3 (Area Bounded by the x-axis): Find the total area of the region bounded by $f(x) = x^2 - 4$ and the x-axis.

Step 1: Find the x-intercepts to determine the limits.
$x^2 - 4 = 0 \Rightarrow x = \pm 2$
Step 2: Determine the sign of $f(x)$ on $[-2, 2]$.
Test $x = 0$: $f(0) = -4 < 0$. The parabola is *below* the x-axis on $(-2, 2)$.
Step 3: Set up the total area integral. Since $f(x) < 0$ on $(-2, 2)$, negate the integral.

$$\text{Total Area} = -\int_{-2}^{2} (x^2 - 4)\,dx$$

$$\int_{-2}^{2} (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$$

At $x = 2$: $\frac{8}{3} - 8 = -\frac{16}{3}$
At $x = -2$: $-\frac{8}{3} + 8 = \frac{16}{3}$

$$-\frac{16}{3} - \frac{16}{3} = -\frac{32}{3}$$

$$\text{Total Area} = -\left(-\frac{32}{3}\right) = \frac{32}{3}$$

$$\boxed{\text{Total Area} = \frac{32}{3}}$$

Example 4 (Curve Crosses the x-axis Multiple Times): Find the total area between $f(x) = \sin(x)$ and the x-axis on $[0, 2\pi]$.

Step 1: Find the zeros of $\sin(x)$ on $[0, 2\pi]$.
$\sin(x) = 0$ at $x = 0, \pi, 2\pi$.
Step 2: Determine the sign on each subinterval.

- On $(0, \pi)$: $\sin(x) > 0$
- On $(\pi, 2\pi)$: $\sin(x) < 0$

Step 3: Set up the total area, negating the negative piece.

$$\text{Total Area} = \int_0^{\pi} \sin(x)\,dx - \int_{\pi}^{2\pi} \sin(x)\,dx$$

$$\int_0^{\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]_0^{\pi} = -\cos(\pi) + \cos(0) = 1 + 1 = 2$$

$$\int_{\pi}^{2\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]_{\pi}^{2\pi} = -\cos(2\pi) + \cos(\pi) = -1 - 1 = -2$$

$$\text{Total Area} = 2 - (-2) = 4$$

$$\boxed{\text{Total Area} = 4}$$

Example 5 (Setting Up from a Word Problem): Find the area of the region enclosed by $f(x) = 3x - x^2$ and the x-axis.

Step 1: Find where the curve meets the x-axis.
$3x - x^2 = 0 \Rightarrow x(3 - x) = 0 \Rightarrow x = 0$ and $x = 3$.
Step 2: Check the sign on $(0, 3)$.
Test $x = 1$: $f(1) = 3 - 1 = 2 > 0$. The curve is above the x-axis on $(0, 3)$.
Step 3: Integrate directly (no negation needed).

$$\int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

At $x = 3$: $\dfrac{27}{2} - 9 = \dfrac{27 - 18}{2} = \dfrac{9}{2}$
At $x = 0$: 0

$$\boxed{\text{Area} = \frac{9}{2}}$$

Common Mistakes

- **Confusing signed and total area:** Evaluating $\int_a^b f(x)\,dx$ directly when f goes negative gives signed area, not total area. If the question asks for the area of a region, always check for sign changes.
- **Forgetting to find all zeros inside the interval:** If f crosses the x-axis more than once inside $[a, b]$, you must split at every crossing. Missing one zero causes positive and negative parts to cancel incorrectly.
- **Not negating the negative piece:** When $f(x) < 0$ on a subinterval, the integral over that piece is negative. You must multiply by -1 to get a positive area contribution. Adding a negative value makes the total area smaller, not larger.
- **Using the wrong limits:** For "bounded by the x-axis" problems, the limits are the x-intercepts, not arbitrary values. Always solve $f(x) = 0$ first.
- **Assuming symmetry without checking:** Just because an interval is symmetric around zero does not mean areas cancel. Check whether f is odd or even before assuming anything about cancellation.

Worksheet

1. Find the area under $f(x) = x^2 + 1$ from $x = 0$ to $x = 3$.

2. Find the signed area $\displaystyle\int_0^2 (x^2 - 3x)\,dx$. Then find the total area between $f(x) = x^2 - 3x$ and the x-axis on $[0, 3]$.

3. Find the total area enclosed by $f(x) = x^2 - 9$ and the x-axis.

4. Find the total area between $f(x) = \cos(x)$ and the x-axis on $[0, 2\pi]$.

5. Find the area enclosed by $f(x) = 4x - x^2$ and the x-axis.

6. Find the total area between $f(x) = x^3 - 4x$ and the x-axis on $[-2, 2]$.

7. Find the area under $f(x) = e^x$ from $x = 0$ to $x = 2$.

8. A particle moves along a line with velocity $v(t) = t^2 - 3t + 2$.

   (a) Find the displacement of the particle from $t = 0$ to $t = 3$.

   (b) Find the total distance traveled from $t = 0$ to $t = 3$.

   (Displacement = signed area; total distance = total area under $|v(t)|$.)

9. (Challenge) Find the total area between $f(x) = x^3 - 3x^2 + 2x$ and the x-axis on $[0, 2]$.

10. (Challenge) Find the total area enclosed between $f(x) = \sin(x) - \cos(x)$ and the x-axis on $[0, \pi]$.

Answer Key

1. $\displaystyle\int_0^3 (x^2+1)\,dx$

$f(x) = x^2 + 1 > 0$ everywhere — no sign changes.

$$\left[\frac{x^3}{3} + x\right]_0^3 = 9 + 3 - 0 = 12$$

$$\boxed{\text{Area} = 12}$$

2. $f(x) = x^2 - 3x$ on $[0, 3]$.

Signed area:

$$\int_0^2 (x^2 - 3x)\,dx = \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^2 = \frac{8}{3} - 6 = \frac{8-18}{3} = -\frac{10}{3}$$

$$\boxed{\text{Signed Area on } [0,2] = -\frac{10}{3}}$$

Total area on $[0,3]$**:** Find zeros: $x(x-3) = 0 \Rightarrow x = 0, 3$. Test $x = 1$: $f(1) = 1 - 3 = -2 < 0$. Curve is below axis on $(0, 3)$.

$$\text{Total Area} = -\int_0^3 (x^2 - 3x)\,dx = -\left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3 = -\left(9 - \frac{27}{2}\right) = -\left(-\frac{9}{2}\right) = \frac{9}{2}$$

$$\boxed{\text{Total Area on } [0,3] = \frac{9}{2}}$$

3. $f(x) = x^2 - 9$. Zeros: $x = \pm 3$. Test $x = 0$: $f(0) = -9 < 0$. Curve is below axis on $(-3, 3)$.

$$\text{Total Area} = -\int_{-3}^3 (x^2 - 9)\,dx = -\left[\frac{x^3}{3} - 9x\right]_{-3}^3$$

At $x = 3$: $9 - 27 = -18$. At $x = -3$: $-9 + 27 = 18$.

$$= -(-18 - 18) = -(-36) = 36$$

$$\boxed{\text{Total Area} = 36}$$

4. $f(x) = \cos(x)$ on $[0, 2\pi]$. Zeros at $x = \pi/2$ and $x = 3\pi/2$.

- $(0, \pi/2)$: $\cos(x) > 0$
- $(\pi/2, 3\pi/2)$: $\cos(x) < 0$
- $(3\pi/2, 2\pi)$: $\cos(x) > 0$

$$\text{Total Area} = \int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{3\pi/2} \cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$$
$$= \Big[\sin x\Big]_0^{\pi/2} - \Big[\sin x\Big]_{\pi/2}^{3\pi/2} + \Big[\sin x\Big]_{3\pi/2}^{2\pi}$$
$$= (1-0) - (-1-1) + (0-(-1)) = 1+2+1 = 4$$
$$\boxed{\text{Total Area} = 4}$$

5. $f(x) = 4x - x^2$. Zeros: $x(4-x) = 0 \Rightarrow x = 0, 4$. Test $x = 2$: $f(2) = 4 > 0$. Above axis.

$$\int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{96-64}{3} = \frac{32}{3}$$
$$\boxed{\text{Area} = \frac{32}{3}}$$

6. $f(x) = x^3 - 4x$ on $[-2, 2]$. Zeros: $x(x^2-4) = 0 \Rightarrow x = -2, 0, 2$.

- On $(-2, 0)$: test $x = -1$: $f(-1) = -1 + 4 = 3 > 0$
- On $(0, 2)$: test $x = 1$: $f(1) = 1 - 4 = -3 < 0$

$$\text{Total Area} = \int_{-2}^{0} (x^3 - 4x)\,dx - \int_0^2 (x^3 - 4x)\,dx$$

Antiderivative: $F(x) = \dfrac{x^4}{4} - 2x^2$.

$$\int_{-2}^{0} = F(0) - F(-2) = 0 - (4-8) = 0 - (-4) = 4$$
$$\int_0^2 = F(2) - F(0) = (4-8) - 0 = -4$$
$$\text{Total Area} = 4 - (-4) = 8$$
$$\boxed{\text{Total Area} = 8}$$

7. $\displaystyle\int_0^2 e^x\,dx = \Big[e^x\Big]_0^2 = e^2 - 1$

$$\boxed{\text{Area} = e^2 - 1}$$

8. $v(t) = t^2 - 3t + 2 = (t-1)(t-2)$.
Zeros at $t = 1$ and $t = 2$. Signs: positive on $(0,1)$, negative on $(1,2)$, positive on $(2,3)$.
(a) Displacement (signed area):

$$\int_0^3 (t^2 - 3t + 2)\,dt = \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_0^3 = 9 - \frac{27}{2} + 6 = 15 - \frac{27}{2} = \frac{3}{2}$$

$$\boxed{\text{Displacement} = \frac{3}{2} \text{ units}}$$

(b) Total distance traveled:

$$\int_0^1 (t^2 - 3t + 2)\, dt = \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_0^1 = \frac{1}{3} - \frac{3}{2} + 2 = \frac{2 - 9 + 12}{6} = \frac{5}{6}$$

$$\int_1^2 (t^2 - 3t + 2)\, dt = \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_1^2 = \left(\frac{8}{3} - 6 + 4\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right) = \frac{2}{3} - \frac{5}{6} = \frac{4 - 5}{6} = -\frac{1}{6}$$

$$\int_2^3 (t^2 - 3t + 2)\, dt = \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_2^3 = \left(9 - \frac{27}{2} + 6\right) - \left(\frac{8}{3} - 6 + 4\right) = \frac{3}{2} - \frac{2}{3} = \frac{9 - 4}{6} = \frac{5}{6}$$

$$\text{Total Distance} = \frac{5}{6} + \left|-\frac{1}{6}\right| + \frac{5}{6} = \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6}$$

$$\boxed{\text{Total Distance} = \frac{11}{6} \text{ units}}$$

9. (Challenge) $f(x) = x^3 - 3x^2 + 2x$ on $[0, 2]$.
Zeros: $x(x^2 - 3x + 2) = x(x-1)(x-2) = 0 \Rightarrow x = 0, 1, 2$.
Signs: test $x = 0.5$: $f(0.5) = 0.125 - 0.75 + 1 = 0.375 > 0$; test $x = 1.5$: $f(1.5) = 3.375 - 6.75 + 3 = -0.375 < 0$.

$$\text{Total Area} = \int_0^1 (x^3 - 3x^2 + 2x)\, dx - \int_1^2 (x^3 - 3x^2 + 2x)\, dx$$

Antiderivative: $F(x) = \dfrac{x^4}{4} - x^3 + x^2$.

$$\int_0^1 = F(1) - F(0) = \frac{1}{4} - 1 + 1 = \frac{1}{4}$$

$$\int_1^2 = F(2) - F(1) = (4 - 8 + 4) - \frac{1}{4} = 0 - \frac{1}{4} = -\frac{1}{4}$$

$$\text{Total Area} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

$$\boxed{\text{Total Area} = \frac{1}{2}}$$

10. (Challenge) $f(x) = \sin(x) - \cos(x)$ on $[0, \pi]$.
Find zeros: $\sin(x) = \cos(x) \Rightarrow \tan(x) = 1 \Rightarrow x = \dfrac{\pi}{4}$ on $[0, \pi]$.
Also check $x = 0$: $f(0) = 0 - 1 = -1 < 0$. And $x = \pi/2$: $f(\pi/2) = 1 - 0 = 1 > 0$.
So $f(x) < 0$ on $(0, \pi/4)$ and $f(x) > 0$ on $(\pi/4, \pi)$.

$$\text{Total Area} = -\int_0^{\pi/4} (\sin x - \cos x)\,dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\,dx$$

Antiderivative: $-\cos(x) - \sin(x)$.

$$\int_0^{\pi/4} = \Big[-\cos x - \sin x\Big]_0^{\pi/4} = \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) - (-1 - 0) = -\sqrt{2} + 1$$

$$-\int_0^{\pi/4} = -(-\sqrt{2} + 1) = \sqrt{2} - 1$$

$$\int_{\pi/4}^{\pi} = \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi} = (-(-1) - 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 1 + \sqrt{2}$$

$$\text{Total Area} = (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}$$

$$\boxed{\text{Total Area} = 2\sqrt{2}}$$

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End of Lesson • ApexMath
