Average Value of a Function
ApexMath

Notes

The Big Idea: What Is the "Average" of a Curve?

You already know how to find the average of a list of numbers. For example, the average of $3, 7, 5, 9$ is:

$$\frac{3+7+5+9}{4} = \frac{24}{4} = 6$$

The pattern is: add everything up, then divide by how many things there are.

Now imagine you want the average value of a *continuous function* $f(x)$ over an interval $[a, b]$. There are infinitely many x-values, so you cannot just add a list. But the same pattern still applies:

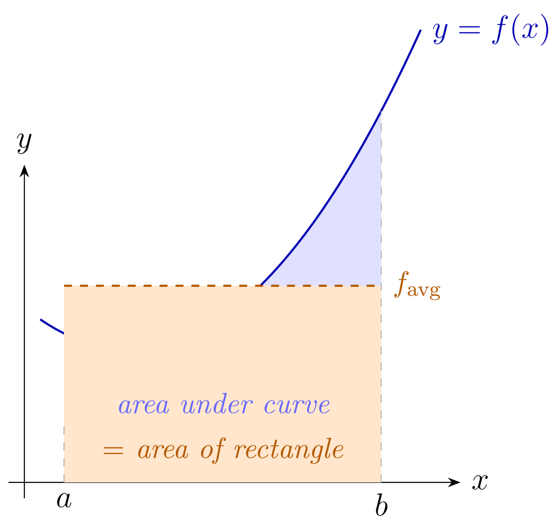
- **"Add everything up"** means integrate: $\displaystyle\int_a^b f(x)\,dx$
- **"Divide by how many"** means divide by the length of the interval: $(b - a)$

This gives the **average value formula**:

$$\boxed{f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Visualizing the Average Value

The average value f_{avg} has a beautiful geometric meaning. It is the height of a horizontal rectangle that has the *same area* as the region under the curve from a to b.



In other words, if you "leveled out" the curve into a flat, constant height, that height would be f_{avg}, and the area would be unchanged.

The Mean Value Theorem for Integrals

There is an important theorem that goes along with this idea. It guarantees that the average value is actually *achieved* somewhere on the interval.

Mean Value Theorem for Integrals: If f is continuous on $[a,b]$, then there exists at least one number c in $[a,b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = f_{\text{avg}}$$

In plain language: a continuous function always hits its average value at least once.

To find c, you follow two steps:

1. Compute f_{avg} using the average value formula.
2. Solve the equation $f(c) = f_{\text{avg}}$ for c, keeping only values in $[a,b]$.

Pro Tip

- **Do not forget to divide by $(b-a)$.** The most common error is computing the integral and stopping there. The average value is the integral *divided* by the length of the interval. Always write $\frac{1}{b-a}$ in front before integrating.
- **$(b-a)$ is the length, not b alone.** If the interval is $[2,7]$, you divide by $7-2=5$, not by 7.
- **When finding c, check that your answer is inside $[a,b]$.** After solving $f(c) = f_{\text{avg}}$, you may get multiple solutions. Discard any that fall outside the interval.
- **The average value can be negative.** If $f(x) < 0$ on most of the interval, then the integral will be negative and so will f_{avg}. This is perfectly valid.
- **Think of it as a rectangle check.** If your answer for f_{avg} seems much larger than the maximum of f on the interval, or smaller than the minimum, you have made an error. The average value must lie between the minimum and maximum of f on $[a,b]$.

Examples

Example 1 (Basic Average Value): Find the average value of $f(x) = x^2$ on $[1,4]$.

Step 1: Identify a, b, and write the formula.
$a = 1$, $b = 4$, so $b - a = 3$.

$$f_{\text{avg}} = \frac{1}{3}\int_1^4 x^2\,dx$$

Step 2: Integrate.

$$f_{\text{avg}} = \frac{1}{3}\left[\frac{x^3}{3}\right]_1^4 = \frac{1}{3}\left(\frac{64}{3} - \frac{1}{3}\right) = \frac{1}{3}\cdot\frac{63}{3} = \frac{63}{9}$$

$$\boxed{f_{\text{avg}} = 7}$$

Example 2 (Average Value with Trig): Find the average value of $f(x) = \sin x$ on $[0, \pi]$.

Step 1: Identify a, b, and write the formula.
$a = 0$, $b = \pi$, so $b - a = \pi$.

$$f_{\text{avg}} = \frac{1}{\pi}\int_0^\pi \sin x\,dx$$

Step 2: Integrate.

$$f_{\text{avg}} = \frac{1}{\pi}\Big[-\cos x\Big]_0^\pi = \frac{1}{\pi}\big[-\cos\pi - (-\cos 0)\big] = \frac{1}{\pi}\big[-(-1) + 1\big] = \frac{1}{\pi}(2)$$

$$\boxed{f_{\text{avg}} = \frac{2}{\pi}}$$

Example 3 (Finding c with the MVT): Find the average value of $f(x) = 3x^2 - 2$ on $[0, 3]$, then find the value(s) of c guaranteed by the Mean Value Theorem for Integrals.

Step 1: Compute f_{avg}.
$a = 0$, $b = 3$, so $b - a = 3$.

$$f_{\text{avg}} = \frac{1}{3}\int_0^3 (3x^2 - 2)\,dx = \frac{1}{3}\left[x^3 - 2x\right]_0^3 = \frac{1}{3}\big[(27 - 6) - 0\big] = \frac{1}{3}(21) = 7$$

Step 2: Solve $f(c) = 7$.

$$3c^2 - 2 = 7$$

$$3c^2 = 9$$

$$c^2 = 3$$

$$c = \pm\sqrt{3}$$

Step 3: Keep only values in $[0, 3]$.

$c = -\sqrt{3} \approx -1.73$ is outside $[0, 3]$, so discard it.
$c = \sqrt{3} \approx 1.73$ is inside $[0, 3]$, so keep it.

$$\boxed{f_{\text{avg}} = 7, \quad c = \sqrt{3}}$$

Example 4 (Average Value with e^x): Find the average value of $f(x) = e^{2x}$ on $[0, 2]$.

Step 1: Identify a, b, and write the formula.
$a = 0$, $b = 2$, so $b - a = 2$.

$$f_{\text{avg}} = \frac{1}{2}\int_0^2 e^{2x}\,dx$$

Step 2: Integrate using the rule $\displaystyle\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$.
Here $k = 2$:

$$f_{\text{avg}} = \frac{1}{2}\left[\frac{e^{2x}}{2}\right]_0^2 = \frac{1}{2}\cdot\frac{1}{2}\left[e^{2x}\right]_0^2 = \frac{1}{4}\left(e^4 - e^0\right) = \frac{1}{4}(e^4 - 1)$$

$$\boxed{f_{\text{avg}} = \frac{e^4 - 1}{4}}$$

Example 5 (Finding c with a Trig Function): Find the value of c on $\left[0, \frac{\pi}{2}\right]$ guaranteed by the Mean Value Theorem for Integrals for $f(x) = \cos x$.

Step 1: Compute f_{avg}.
$a = 0$, $b = \frac{\pi}{2}$, so $b - a = \frac{\pi}{2}$.

$$f_{\text{avg}} = \frac{1}{\pi/2}\int_0^{\pi/2} \cos x\,dx = \frac{2}{\pi}\Big[\sin x\Big]_0^{\pi/2} = \frac{2}{\pi}(1 - 0) = \frac{2}{\pi}$$

Step 2: Solve $f(c) = \frac{2}{\pi}$.

$$\cos c = \frac{2}{\pi}$$

$$c = \arccos\left(\frac{2}{\pi}\right)$$

Step 3: Verify c is in $\left[0, \frac{\pi}{2}\right]$.
Since $0 < \frac{2}{\pi} < 1$, the value $\arccos\left(\frac{2}{\pi}\right)$ lies in $\left(0, \frac{\pi}{2}\right)$. ✓

$$\boxed{c = \arccos\left(\frac{2}{\pi}\right) \approx 0.876}$$

Common Mistakes

- **Forgetting to divide by $(b-a)$:** Computing only the integral gives you the net area, not the average value. The division step is what converts area into average height.
- **Dividing by b instead of $(b-a)$:** Always subtract. On $[2,5]$ you divide by 3, not by 5.
- **Not checking that c is inside $[a,b]$:** When solving $f(c) = f_{\text{avg}}$ you may get extra solutions. Always verify each candidate lies within the interval before accepting it.
- **Confusing average value with average rate of change:** The average value is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. The average rate of change is $\dfrac{f(b)-f(a)}{b-a}$. These are different things. This lesson is about the first one.
- **Assuming c is the midpoint of $[a,b]$:** The value c where $f(c) = f_{\text{avg}}$ must be found by solving an equation. It is almost never simply $\dfrac{a+b}{2}$.

Worksheet

1. Find the average value of $f(x) = 4x + 1$ on $[0, 3]$.

2. Find the average value of $f(x) = x^3$ on $[-2, 2]$.

3. Find the average value of $f(x) = \cos x$ on $[0, \pi]$.

4. Find the average value of $f(x) = e^x$ on $[0, \ln 4]$.

5. Find the average value of $f(x) = \dfrac{1}{x}$ on $[1, e]$.

6. Find the average value of $f(x) = x^2 - 3x$ on $[1, 4]$, then find all values of c in $[1, 4]$ guaranteed by the Mean Value Theorem for Integrals.

7. Find the average value of $f(x) = \sin x$ on $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$.

8. (Challenge) The temperature in a city over a 12-hour period is modeled by $T(t) = 60 + 10\sin\left(\frac{\pi t}{12}\right)$ degrees Fahrenheit, where t is hours after midnight. Find the average temperature from $t = 0$ to $t = 12$.

Answer Key

1. Average value of $f(x) = 4x + 1$ on $[0, 3]$.

Step 1: $b - a = 3 - 0 = 3$. Write the formula.

$$f_{\text{avg}} = \frac{1}{3}\int_0^3 (4x+1)\,dx$$

Step 2: Integrate.

$$f_{\text{avg}} = \frac{1}{3}\left[2x^2 + x\right]_0^3 = \frac{1}{3}\big[(18+3) - 0\big] = \frac{1}{3}(21)$$

$$\boxed{f_{\text{avg}} = 7}$$

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2. Average value of $f(x) = x^3$ on $[-2, 2]$.

Step 1: $b - a = 2 - (-2) = 4$. Write the formula.

$$f_{\text{avg}} = \frac{1}{4}\int_{-2}^2 x^3\,dx$$

Step 2: Notice that $f(x) = x^3$ is an odd function and the interval is symmetric about 0. The integral of any odd function over a symmetric interval $[-a, a]$ equals zero:

$$\int_{-2}^2 x^3\,dx = 0$$

Step 3: Conclude.

$$f_{\text{avg}} = \frac{1}{4}(0)$$

$$\boxed{f_{\text{avg}} = 0}$$

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3. Average value of $f(x) = \cos x$ on $[0, \pi]$.

Step 1: $b - a = \pi - 0 = \pi$. Write the formula.

$$f_{\text{avg}} = \frac{1}{\pi}\int_0^\pi \cos x\,dx$$

Step 2: Integrate.

$$f_{\text{avg}} = \frac{1}{\pi}\Big[\sin x\Big]_0^\pi = \frac{1}{\pi}(\sin\pi - \sin 0) = \frac{1}{\pi}(0 - 0)$$

$$\boxed{f_{\text{avg}} = 0}$$

Note: This makes sense geometrically — $\cos x$ is positive on $[0, \pi/2]$ and equally negative on $[\pi/2, \pi]$, so the areas cancel and the average is zero.

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4. Average value of $f(x) = e^x$ on $[0, \ln 4]$.

Step 1: $b - a = \ln 4 - 0 = \ln 4$. Write the formula.

$$f_{\text{avg}} = \frac{1}{\ln 4} \int_0^{\ln 4} e^x \, dx$$

Step 2: Integrate.

$$f_{\text{avg}} = \frac{1}{\ln 4} \Big[e^x \Big]_0^{\ln 4} = \frac{1}{\ln 4} \left(e^{\ln 4} - e^0 \right) = \frac{1}{\ln 4}(4 - 1) = \frac{3}{\ln 4}$$

$$\boxed{f_{\text{avg}} = \frac{3}{\ln 4}}$$

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5. Average value of $f(x) = \dfrac{1}{x}$ on $[1, e]$.

Step 1: $b - a = e - 1$. Write the formula.

$$f_{\text{avg}} = \frac{1}{e-1} \int_1^e \frac{1}{x} \, dx$$

Step 2: Integrate. Recall $\displaystyle\int \frac{1}{x} \, dx = \ln|x| + C$.

$$f_{\text{avg}} = \frac{1}{e-1} \Big[\ln x \Big]_1^e = \frac{1}{e-1}(\ln e - \ln 1) = \frac{1}{e-1}(1 - 0)$$

$$\boxed{f_{\text{avg}} = \frac{1}{e-1}}$$

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6. Average value of $f(x) = x^2 - 3x$ on $[1, 4]$, and find c.

Step 1: $b - a = 4 - 1 = 3$. Write the formula.

$$f_{\text{avg}} = \frac{1}{3} \int_1^4 (x^2 - 3x) \, dx$$

Step 2: Integrate.

$$f_{\text{avg}} = \frac{1}{3} \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_1^4$$

At $x = 4$: $\dfrac{64}{3} - \dfrac{48}{2} = \dfrac{64}{3} - 24 = \dfrac{64 - 72}{3} = -\dfrac{8}{3}$

At $x = 1$: $\dfrac{1}{3} - \dfrac{3}{2} = \dfrac{2 - 9}{6} = -\dfrac{7}{6}$

$$f_{\text{avg}} = \frac{1}{3}\left(-\frac{8}{3} - \left(-\frac{7}{6}\right)\right) = \frac{1}{3}\left(-\frac{16}{6} + \frac{7}{6}\right) = \frac{1}{3}\left(-\frac{9}{6}\right) = \frac{1}{3}\left(-\frac{3}{2}\right) = -\frac{1}{2}$$

Step 3: Solve $f(c) = -\dfrac{1}{2}$.

$$c^2 - 3c = -\frac{1}{2}$$

Multiply through by 2 to clear the fraction:

$$2c^2 - 6c = -1$$

$$2c^2 - 6c + 1 = 0$$

Use the quadratic formula with $a = 2$, $b = -6$, $d = 1$:

$$c = \frac{6 \pm \sqrt{36 - 8}}{4} = \frac{6 \pm \sqrt{28}}{4} = \frac{6 \pm 2\sqrt{7}}{4} = \frac{3 \pm \sqrt{7}}{2}$$

Step 4: Check which values lie in $[1, 4]$.

$c = \dfrac{3 + \sqrt{7}}{2} \approx \dfrac{3 + 2.646}{2} \approx 2.82$ — inside $[1, 4]$. ✓

$c = \dfrac{3 - \sqrt{7}}{2} \approx \dfrac{3 - 2.646}{2} \approx 0.18$ — outside $[1, 4]$. Discard.

$$\boxed{f_{\text{avg}} = -\frac{1}{2}, \quad c = \frac{3 + \sqrt{7}}{2}}$$

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7. Average value of $f(x) = \sin x$ on $\left[\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right]$.

Step 1: $b - a = \dfrac{5\pi}{6} - \dfrac{\pi}{6} = \dfrac{4\pi}{6} = \dfrac{2\pi}{3}$.

Write the formula.

$$f_{\text{avg}} = \frac{1}{2\pi/3} \int_{\pi/6}^{5\pi/6} \sin x \, dx = \frac{3}{2\pi} \int_{\pi/6}^{5\pi/6} \sin x \, dx$$

Step 2: Integrate.

$$f_{\text{avg}} = \frac{3}{2\pi} \Big[-\cos x \Big]_{\pi/6}^{5\pi/6} = \frac{3}{2\pi} \left(-\cos\frac{5\pi}{6} + \cos\frac{\pi}{6} \right)$$

Recall $\cos\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{2}$ and $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$:

$$f_{\text{avg}} = \frac{3}{2\pi} \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \right) = \frac{3}{2\pi} \cdot \sqrt{3} = \frac{3\sqrt{3}}{2\pi}$$

$$\boxed{f_{\text{avg}} = \frac{3\sqrt{3}}{2\pi}}$$

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8. (Challenge) $T(t) = 60 + 10\sin\left(\frac{\pi t}{12}\right)$, average on $[0, 12]$.

Step 1: $b - a = 12 - 0 = 12$. Write the formula.

$$T_{\text{avg}} = \frac{1}{12}\int_0^{12}\left(60 + 10\sin\frac{\pi t}{12}\right)dt$$

Step 2: Split the integral into two parts.

$$T_{\text{avg}} = \frac{1}{12}\int_0^{12} 60\,dt + \frac{1}{12}\int_0^{12} 10\sin\frac{\pi t}{12}\,dt$$

Step 3: Evaluate the first part.

$$\frac{1}{12}\int_0^{12} 60\,dt = \frac{1}{12}\cdot 60\cdot 12 = 60$$

Step 4: Evaluate the second part.
Use the rule $\int \sin(kt)\,dt = -\frac{\cos(kt)}{k} + C$ with $k = \frac{\pi}{12}$:

$$\frac{10}{12}\int_0^{12}\sin\frac{\pi t}{12}\,dt = \frac{10}{12}\left[-\frac{12}{\pi}\cos\frac{\pi t}{12}\right]_0^{12} = \frac{10}{12}\cdot\left(-\frac{12}{\pi}\right)\left[\cos\frac{\pi t}{12}\right]_0^{12}$$

$$= -\frac{10}{\pi}\left(\cos\pi - \cos 0\right) = -\frac{10}{\pi}\left(-1 - 1\right) = -\frac{10}{\pi}(-2) = \frac{20}{\pi}$$

Step 5: Combine.

$$T_{\text{avg}} = 60 + \frac{20}{\pi}$$

$$\boxed{T_{\text{avg}} = 60 + \frac{20}{\pi} \approx 66.4°\text{F}}$$

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