Lesson 4: Chain Rule

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Notes

- **What is the Chain Rule?** The chain rule is a formula to find the derivative of a composition of functions. If $y = f(g(x))$, we call f the *outer function* and g the *inner function*.
- **Chain Rule formulas:** There are two equivalent ways to write it:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{or} \quad f'(x) = f'(g(x)) \cdot g'(x)$$

  Here, $u = g(x)$ is the inner function. First, differentiate the outer function with respect to the inner function, then multiply by the derivative of the inner function.
- **Step 1: Identify outer and inner functions** Example: $y = (3x+2)^5$ - Outer function: $f(u) = u^5$ - Inner function: $u = 3x + 2$
- **Step 2: Differentiate each part separately** - Using $dy/du \cdot du/dx$ notation:

$$\frac{dy}{du} = 5u^4, \quad \frac{du}{dx} = 3$$

  - Using $f'(g(x)) \cdot g'(x)$ notation:

$$f'(g(x)) = 5(3x+2)^4, \quad g'(x) = 3$$

- **Step 3: Multiply the derivatives**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5(3x+2)^4 \cdot 3$$

  or equivalently,

$$f'(x) = f'(g(x)) \cdot g'(x) = 5(3x+2)^4 \cdot 3$$

- **Step 4: Simplify**

$$f'(x) = 15(3x+2)^4$$

- **Radicals and exponents:** If the function contains a square root or cube root, rewrite it as a fractional exponent first:

$$\sqrt{x} = x^{1/2}, \quad \sqrt[3]{x} = x^{1/3}$$

  Then apply the chain rule normally.
- **Why it works:** The chain rule accounts for how the inner function $g(x)$ changes and how that affects the outer function $f(u)$. Multiplying dy/du by du/dx or $f'(g(x))$ by $g'(x)$ combines these rates of change.

Pro Tips

- Always identify and label outer and inner functions.
- Differentiate step by step and write each derivative separately.
- Rewriting radicals as exponents makes differentiation easier.
- Check your final simplification carefully.
- For product + chain rule problems (bonus), apply each rule one at a time.

Examples

Example 1: $f(x) = (3x + 2)^5$

Step 1: Outer and inner functions: $y = u^5,\ u = 3x + 2$

Step 2: Derivatives:

$$\frac{dy}{du} = 5u^4, \quad \frac{du}{dx} = 3$$

Step 3: Apply chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5(3x + 2)^4 \cdot 3$$

Step 4: Simplify:

$$f'(x) = 15(3x + 2)^4$$

$$\boxed{\text{f'(x) = 15(3x+2)}^4}$$

Example 2: $f(x) = \sqrt{2x + 1}$

Step 1: Rewrite radical: $y = u^{1/2},\ u = 2x + 1$

Step 2: Derivatives:

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}, \quad \frac{du}{dx} = 2$$

Step 3: Apply chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}(2x + 1)^{-1/2} \cdot 2$$

Step 4: Simplify:

$$f'(x) = (2x + 1)^{-1/2} = \frac{1}{\sqrt{2x + 1}}$$

$$\boxed{\text{f'(x) = 1/}\sqrt{2x + 1}}$$

Common Mistakes

- Forgetting to multiply by the derivative of the inner function.
- Confusing outer and inner functions.
- Skipping steps when simplifying exponents or radicals.
- Not rewriting radicals before differentiating.
- Applying the chain rule more than once unnecessarily.

Worksheet Problems

1. $f(x) = (4x+1)^3$
2. $f(x) = \sqrt{5x-3}$
3. $f(x) = (x^2+1)^4$
4. $f(x) = \sqrt[3]{3x+2}$
5. Bonus: $f(x) = (x+1)(x-2)^3$

Answer Key

1. $f(x) = (4x+1)^3$

Step 1: Identify outer and inner functions:

$$\text{Outer: } f(u) = u^3, \quad \text{Inner: } u = 4x+1$$

Step 2: Differentiate each part:

$$\frac{dy}{du} = 3u^2, \quad \frac{du}{dx} = 4$$

Step 3: Apply chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3(4x+1)^2 \cdot 4$$

Step 4: Simplify:

$$f'(x) = 12(4x+1)^2$$

$$\boxed{\text{f'(x) = 12(4x+1)}^2}$$

2. $f(x) = \sqrt{5x-3}$

Step 1: Rewrite radical as exponent and identify outer/inner functions:

$$y = u^{1/2}, \quad u = 5x-3$$

Step 2: Differentiate:

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}, \quad \frac{du}{dx} = 5$$

Step 3: Apply chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}(5x-3)^{-1/2} \cdot 5$$

Step 4: Simplify:

$$f'(x) = \frac{5}{2}(5x-3)^{-1/2} = \frac{5}{2\sqrt{5x-3}}$$

$$\boxed{\text{f'(x) = }5/(2\sqrt{5x-3})}$$

3. $f(x) = (x^2+1)^4$

Step 1: Identify outer and inner functions:

$$f(u) = u^4, \quad u = x^2+1$$

Step 2: Differentiate each part:

$$\frac{dy}{du} = 4u^3, \quad \frac{du}{dx} = 2x$$

Step 3: Apply chain rule:

$$\frac{dy}{dx} = 4(x^2+1)^3 \cdot 2x$$

Step 4: Simplify:

$$f'(x) = 8x(x^2+1)^3$$

$$\boxed{\text{f'(x) = 8x(x}^2 + 1)^3}$$

4. $f(x) = \sqrt[3]{3x+2}$

Step 1: Rewrite radical as exponent and identify functions:

$$y = u^{1/3}, \quad u = 3x+2$$

Step 2: Differentiate each part:

$$\frac{dy}{du} = \frac{1}{3}u^{-2/3}, \quad \frac{du}{dx} = 3$$

Step 3: Apply chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{3}(3x+2)^{-2/3} \cdot 3$$

Step 4: Simplify:

$$f'(x) = (3x+2)^{-2/3} = \frac{1}{\sqrt[3]{(3x+2)^2}}$$

$$\boxed{\text{f'(x)} = 1/\sqrt[3]{(3x+2)^2}}$$

Bonus. $f(x) = (x+1)(x-2)^3$

Step 1: Identify functions for product rule:

$$u = x + 1, \quad v = (x-2)^3$$

Step 2: Differentiate each part:

$$u' = 1, \quad v' = 3(x-2)^2$$

Step 3: Apply product rule:

$$\frac{dy}{dx} = u'v + uv' = 1 \cdot (x-2)^3 + (x+1) \cdot 3(x-2)^2$$

Step 4: Factor common terms:

$$f'(x) = (x-2)^2\big((x-2) + 3(x+1)\big)$$

Step 5: Simplify inside parentheses:

$$(x-2) + 3(x+1) = 4x + 1$$

Step 6: Final derivative:

$$f'(x) = (x-2)^2(4x+1)$$

$$\boxed{\text{f'(x)} = \text{(x-2)}^2(4x+1)}$$

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