Lesson 11: Critical Points and Concavity

ApexMath

Notes

- A **critical point** occurs where the derivative of a function is zero or undefined:

$$f'(x) = 0 \quad \text{or} \quad f'(x) \text{ does not exist.}$$

- Why $f'(x) = 0$ indicates a critical point:
  - The derivative represents the slope of the tangent line.
  - If $f'(x) = 0$, the slope is horizontal.
  - Horizontal tangents often correspond to local maxima, minima, or flat points.
- The **second derivative** tells us about concavity:
  - $f''(x) > 0 \implies$ the graph is concave up (like a cup), so a critical point is a **local minimum**.
  - $f''(x) < 0 \implies$ the graph is concave down (like an upside-down cup), so a critical point is a **local maximum**.
  - $f''(x) = 0 \implies$ test is inconclusive; may need further analysis.
- Steps to analyze a function:
  1. Compute $f'(x)$ and solve $f'(x) = 0$ to find critical points.
  2. Compute $f''(x)$ and evaluate at each critical point to determine concavity.
  3. Identify whether each critical point is a maximum, minimum, or inconclusive.

Pro Tips

- Always simplify derivatives before solving $f'(x) = 0$.
- Sketching the function helps visualize critical points and concavity.
- Remember: endpoints of the domain can also give maxima or minima.
- Use the second derivative test carefully; if it fails, use the first derivative test.

Examples

Example 1: Find and classify the critical points of $f(x) = x^2 - 4x + 3$.
Step 1 – Find the derivative:

$$f'(x) = 2x - 4$$

Step 2 – Solve for critical points:

$$2x - 4 = 0 \implies x = 2$$

Step 3 – Find the second derivative:

$$f''(x) = 2$$

Step 4 – Determine concavity:

$$f''(2) = 2 > 0 \implies \text{concave up, local minimum}$$

Step 5 – Answer:

$$\boxed{x = 2 \text{ is a local minimum}}$$

Example 2: Find and classify the critical points of $f(x) = -x^2 + 6x - 5$.
Step 1 – Derivative:

$$f'(x) = -2x + 6$$

Step 2 – Solve for critical points:

$$-2x + 6 = 0 \implies x = 3$$

Step 3 – Second derivative:

$$f''(x) = -2$$

Step 4 – Determine concavity:

$$f''(3) = -2 < 0 \implies \text{concave down, local maximum}$$

Step 5 – Answer:

$$\boxed{x = 3 \text{ is a local maximum}}$$

Example 3: Analyze $f(x) = x^3$.
Step 1 – Derivative:

$$f'(x) = 3x^2$$

Step 2 – Critical points:

$$3x^2 = 0 \implies x = 0$$

Step 3 – Second derivative:

$$f''(x) = 6x$$

Step 4 – Evaluate concavity at critical point:

$$f''(0) = 0 \implies \text{second derivative test inconclusive, inflection point likely}$$

Step 5 – Answer:

$$\boxed{x = 0 \text{ is an inflection point; neither max nor min}}$$

Example 4 (Higher-degree polynomial): Find and classify the critical points of $f(x) = x^4 - 4x^3 + 6x^2$.

Step 1 – Derivative:

$$f'(x) = 4x^3 - 12x^2 + 12x = 4x(x^2 - 3x + 3)$$

Step 2 – Solve for critical points:

$$4x(x^2 - 3x + 3) = 0 \implies x = 0 \quad \text{or solve } x^2 - 3x + 3 = 0$$

Step 3 – Check discriminant for quadratic:

$$\Delta = (-3)^2 - 4(1)(3) = 9 - 12 = -3 < 0$$

$\implies$ no real solutions from quadratic.

Step 4 – Second derivative:

$$f''(x) = 12x^2 - 24x + 12$$

Step 5 – Evaluate concavity at $x = 0$:

$$f''(0) = 12 > 0 \implies \text{local minimum at } x = 0$$

Step 6 – Answer:

$$\boxed{x = 0 \text{ is a local minimum; no other real critical points}}$$

Common Mistakes

- Forgetting that $f'(x) = 0$ or undefined gives critical points.
- Misclassifying a point without checking $f''(x)$ or concavity.
- Ignoring endpoints when finding global maxima/minima.
- Forgetting that $f''(x) = 0$ means the test is inconclusive.
- Incorrect algebra when solving $f'(x) = 0$.

Worksheet

1. Find and classify the critical points of $f(x) = x^2 - 8x + 12$.
2. Find and classify the critical points of $f(x) = -2x^2 + 8x - 3$.
3. Find and classify the critical points of $f(x) = x^3 - 6x^2 + 9x$.
4. Find and classify the critical points of $f(x) = x^4 - 4x^2$.
5. Find and classify the critical points of $f(x) = x/(x^2 + 1)$.

Answer Key

1. **Question 1:** $f(x) = x^2 - 8x + 12$
   Step 1 – Derivative:
   $$f'(x) = 2x - 8$$
   Step 2 – Solve for critical point:
   $$2x - 8 = 0 \implies x = 4$$
   Step 3 – Second derivative:
   $$f''(x) = 2$$
   Step 4 – Determine concavity: $f''(4) = 2 > 0 \implies$ graph is concave up. **Step 5 – Classification:**
   $$\boxed{x = 4 \text{ is a local minimum}}$$

Question 2: $f(x) = -2x^2 + 8x - 3$
Step 1 – Derivative:
$$f'(x) = -4x + 8$$
Step 2 – Solve for critical point:
$$-4x + 8 = 0 \implies x = 2$$
Step 3 – Second derivative:
$$f''(x) = -4$$
Step 4 – Determine concavity: $f''(2) = -4 < 0 \implies$ graph is concave down. **Step 5 – Classification:**
$$\boxed{x = 2 \text{ is a local maximum}}$$

Question 3: $f(x) = x^3 - 6x^2 + 9x$

Step 1 – Derivative:

$$f'(x) = 3x^2 - 12x + 9$$

Step 2 – Solve for critical points:

$$3x^2 - 12x + 9 = 0 \implies x^2 - 4x + 3 = 0 \implies (x-1)(x-3) = 0$$

$$\implies x = 1, 3$$

Step 3 – Second derivative:

$$f''(x) = 6x - 12$$

Step 4 – Evaluate concavity at each critical point:

$$f''(1) = 6(1) - 12 = -6 < 0 \implies x = 1 \text{ max}$$

$$f''(3) = 6(3) - 12 = 6 > 0 \implies x = 3 \text{ min}$$

Step 5 – Answer:

$$\boxed{x = 1 \text{ max},\ x = 3 \text{ min}}$$

Question 4: $f(x) = x^4 - 4x^2$

Step 1 – Derivative:

$$f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$$

Step 2 – Solve for critical points:

$$4x(x^2 - 2) = 0 \implies x = 0, x = \pm\sqrt{2}$$

Step 3 – Second derivative:

$$f''(x) = 12x^2 - 8$$

Step 4 – Evaluate concavity at each critical point:

$$f''(0) = -8 < 0 \implies x = 0 \text{ max}$$

$$f''(\sqrt{2}) = 16 > 0 \implies x = \sqrt{2} \text{ min}, \quad f''(-\sqrt{2}) = 16 > 0 \implies x = -\sqrt{2} \text{ min}$$

Step 5 – Answer:

$$\boxed{x = 0 \text{ max},\ x = \pm\sqrt{2} \text{ min}}$$

Question 5: $f(x) = \frac{x}{x^2+1}$

Step 1 – Derivative:

$$f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2}$$

Step 2 – Solve for critical points:

$$1 - x^2 = 0 \implies x = \pm 1$$

Step 3 – Second derivative:

$$f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$$

Step 4 – Evaluate concavity:

$$f''(1) = -0.5 < 0 \implies x = 1 \text{ max}, \quad f''(-1) = 0.5 > 0 \implies x = -1 \text{ min}$$

Step 5 – Answer:

$$\boxed{x = 1 \text{ max},\ x = -1 \text{ min}}$$

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