Derivatives Unit Cumulative Practice Test
ApexMath

Part 1: The Definition of a Derivative

1. Using the **limit definition of the derivative**, find $f'(x)$ for $f(x) = x^2 + 3x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Using the limit definition, find $f'(x)$ for $f(x) = 4x - 7$.

3. Explain in your own words what the derivative $f'(a)$ represents geometrically on the graph of $f(x)$.

Part 2: Basic Derivative Rules

Differentiate each function. Show all steps.

4. $f(x) = 7x^5 - 3x^3 + 2x - 9$

5. $g(x) = \sqrt{x} + \frac{1}{x^2}$

6. $h(x) = 4x^{3/2} - 6x^{-1} + 5$

7. $f(x) = \pi^3 + x^4 - 8x$

Part 3: Product and Quotient Rule

8. $f(x) = (3x^2 + 1)(x^3 - 5x)$ *(Use the product rule)*

9. $g(x) = x^4 \cdot \sin(x)$ *(Use the product rule)*

10. $h(x) = \dfrac{x^2 + 4}{x - 1}$ *(Use the quotient rule)*

11. $f(x) = \dfrac{e^x}{x^3}$ *(Use the quotient rule)*

Part 4: Chain Rule

12. $f(x) = (5x^2 - 3)^4$

13. $g(x) = \sqrt{4x + 7}$

14. $h(x) = \sin(3x^2)$

15. $f(x) = e^{2x^3 - x}$

Part 5: Derivatives of Exponential and Logarithmic Functions

16. $f(x) = e^{5x}$

17. $g(x) = 3^x$

18. $h(x) = \ln(x^2 + 1)$

19. $f(x) = x^2 \cdot e^x$

20. $g(x) = \ln(\sin(x))$

Part 6: Trigonometric and Inverse Trigonometric Derivatives

21. $f(x) = \sin(x) + \cos(x)$

22. $g(x) = \tan(x) - \sec(x)$

23. $h(x) = x^2 \cos(x)$ *(Product rule + trig derivative)*

24. $f(x) = \arcsin(x)$

25. $g(x) = \arctan(3x)$

Part 7: Derivatives of Cotangent, Secant, and Cosecant

26. $f(x) = \cot(x)$

27. $g(x) = \sec(x) \cdot \tan(x)$ *(Product rule)*

28. $h(x) = \csc(4x)$ *(Chain rule)*

Part 8: Higher Order Derivatives

29. Find $f''(x)$ for $f(x) = 5x^4 - 3x^2 + 2x$.

30. Find the third derivative $f'''(x)$ for $f(x) = e^{2x}$.

31. Find $f''(x)$ for $f(x) = x\sin(x)$.

Part 9: Implicit Differentiation

32. Find $\dfrac{dy}{dx}$ for $x^2 + y^2 = 25$.

33. Find $\dfrac{dy}{dx}$ for $x^3 + y^3 = 6xy$.

34. Find $\dfrac{dy}{dx}$ for $\sin(xy) = x$.

Part 10: Related Rates

35. A spherical balloon is being inflated. The radius is increasing at a rate of 2 cm/s. How fast is the volume increasing when the radius is 5 cm?

$$\textit{Recall: } V = \tfrac{4}{3}\pi r^3$$

36. A 10-foot ladder is leaning against a wall. The base of the ladder slides away from the wall at 1 ft/s. How fast is the top of the ladder sliding down the wall when the base is 6 ft from the wall?

Part 11: Critical Points and Concavity

37. For $f(x) = x^3 - 6x^2 + 9x + 1$:

    (a) Find all critical points.

    (b) Use the first derivative test to classify each as a local max, local min, or neither.

    (c) Find the intervals where f is concave up and concave down.

    (d) Find any inflection points.

38. For $g(x) = x^4 - 8x^2$, find all critical points and use the **second derivative test** to classify them.

Part 12: Real-World Optimization Problems

39. A farmer has 200 meters of fencing and wants to enclose a rectangular field with one side along a barn (no fencing needed on that side). What dimensions maximize the enclosed area?

40. A cylindrical can must hold 500π cm^3 of volume. Find the radius and height that minimize the total surface area.

$$\textit{Recall: } V = \pi r^2 h, \quad SA = 2\pi r^2 + 2\pi rh$$

Part 13: Derivatives in Physics

41. The position of a particle is given by $s(t) = t^3 - 6t^2 + 9t$, where s is in meters and t is in seconds.

    (a) Find the velocity function $v(t)$.

    (b) Find the acceleration function $a(t)$.

    (c) At what times is the particle at rest (velocity $= 0$)?

    (d) Is the particle speeding up or slowing down at $t = 4$?

Part 14: Linearization and Linear Approximation

42. Find the linearization $L(x)$ of $f(x) = \sqrt{x}$ at $a = 16$. Use it to approximate $\sqrt{16.5}$.

43. Find the linearization of $f(x) = e^x$ at $a = 0$. Use it to approximate $e^{-0.1}$.

Part 15: Newton's Method

44. Use Newton's Method with $x_0 = 2$ to approximate a root of $f(x) = x^2 - 6$. Perform **three** iterations.

45. No starting guess is given. Use Newton's Method to approximate a root of $f(x) = x^3 + x - 5$. Choose your own x_0 by finding a sign change, then perform **three** iterations.

Answer Key: Derivatives Unit Cumulative Practice Test

Part 1: The Definition of a Derivative

1. $f(x) = x^2 + 3x$, find $f'(x)$ using the limit definition.
Step 1: Write the definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Find $f(x+h)$.

$$f(x+h) = (x+h)^2 + 3(x+h) = x^2 + 2xh + h^2 + 3x + 3h$$

Step 3: Subtract $f(x) = x^2 + 3x$.

$$f(x+h) - f(x) = x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x = 2xh + h^2 + 3h$$

Step 4: Divide by h.

$$\frac{2xh + h^2 + 3h}{h} = 2x + h + 3$$

Step 5: Take the limit as $h \to 0$.

$$f'(x) = \lim_{h \to 0}(2x + h + 3) = 2x + 3$$

$$\boxed{f'(x) = 2x + 3}$$

2. $f(x) = 4x - 7$, find $f'(x)$ using the limit definition.
Step 1: Find $f(x+h)$.

$$f(x+h) = 4(x+h) - 7 = 4x + 4h - 7$$

Step 2: Subtract $f(x) = 4x - 7$.

$$f(x+h) - f(x) = 4x + 4h - 7 - 4x + 7 = 4h$$

Step 3: Divide by h.

$$\frac{4h}{h} = 4$$

Step 4: Take the limit.

$$f'(x) = \lim_{h \to 0} 4 = 4$$

$$\boxed{f'(x) = 4}$$

3. Geometric meaning of $f'(a)$.
The derivative $f'(a)$ represents the **slope of the tangent line** to the graph of $f(x)$ at the point $(a, f(a))$. It tells us how steeply the function is rising or falling at exactly that point.

Part 2: Basic Derivative Rules

4. $f(x) = 7x^5 - 3x^3 + 2x - 9$

Apply the power rule to each term. The derivative of a constant is zero.

$$f'(x) = 7 \cdot 5x^4 - 3 \cdot 3x^2 + 2 \cdot 1 - 0$$

$$\boxed{f'(x) = 35x^4 - 9x^2 + 2}$$

5. $g(x) = \sqrt{x} + \dfrac{1}{x^2}$

Step 1: Rewrite using exponents.

$$g(x) = x^{1/2} + x^{-2}$$

Step 2: Apply the power rule to each term.

$$g'(x) = \frac{1}{2}x^{-1/2} + (-2)x^{-3}$$

$$\boxed{g'(x) = \frac{1}{2\sqrt{x}} - \frac{2}{x^3}}$$

6. $h(x) = 4x^{3/2} - 6x^{-1} + 5$

$$h'(x) = 4 \cdot \frac{3}{2}x^{1/2} - 6 \cdot (-1)x^{-2} + 0 = 6x^{1/2} + 6x^{-2}$$

$$\boxed{h'(x) = 6\sqrt{x} + \frac{6}{x^2}}$$

7. $f(x) = \pi^3 + x^4 - 8x$

Note: π^3 is a constant, so its derivative is zero.

$$f'(x) = 0 + 4x^3 - 8$$

$$\boxed{f'(x) = 4x^3 - 8}$$

Part 3: Product and Quotient Rule

8. $f(x) = (3x^2 + 1)(x^3 - 5x)$

Product rule: $(uv)' = u'v + uv'$. Let $u = 3x^2 + 1$ and $v = x^3 - 5x$.

$$u' = 6x \qquad v' = 3x^2 - 5$$

$$\begin{aligned} f'(x) &= (6x)(x^3 - 5x) + (3x^2 + 1)(3x^2 - 5) \\ &= 6x^4 - 30x^2 + 9x^4 - 15x^2 + 3x^2 - 5 \\ &= 15x^4 - 42x^2 - 5 \end{aligned}$$

$$\boxed{f'(x) = 15x^4 - 42x^2 - 5}$$

9. $g(x) = x^4 \cdot \sin(x)$

Let $u = x^4$ and $v = \sin(x)$.

$$u' = 4x^3 \qquad v' = \cos(x)$$

$$g'(x) = 4x^3 \sin(x) + x^4 \cos(x)$$

$$\boxed{g'(x) = 4x^3 \sin(x) + x^4 \cos(x)}$$

10. $h(x) = \dfrac{x^2 + 4}{x - 1}$

Quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$. Let $u = x^2 + 4$ and $v = x - 1$.

$$u' = 2x \qquad v' = 1$$

$$h'(x) = \frac{(2x)(x-1) - (x^2+4)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 4}{(x-1)^2} = \frac{x^2 - 2x - 4}{(x-1)^2}$$

$$\boxed{h'(x) = \frac{x^2 - 2x - 4}{(x-1)^2}}$$

11. $f(x) = \dfrac{e^x}{x^3}$

Let $u = e^x$ and $v = x^3$.

$$u' = e^x \qquad v' = 3x^2$$

$$\begin{aligned} f'(x) &= \frac{e^x \cdot x^3 - e^x \cdot 3x^2}{x^6} = \frac{e^x(x^3 - 3x^2)}{x^6} = \frac{e^x \cdot x^2(x-3)}{x^6} \\ &= \frac{e^x(x-3)}{x^4} \end{aligned}$$

$$\boxed{f'(x) = \frac{e^x(x-3)}{x^4}}$$

Part 4: Chain Rule

12. $f(x) = (5x^2 - 3)^4$
Chain rule: $\frac{d}{dx}[g(h(x))] = g'(h(x)) \cdot h'(x)$.
Outer function: $(\,\cdot\,)^4$, inner function: $5x^2 - 3$.

$$f'(x) = 4(5x^2 - 3)^3 \cdot 10x = 40x(5x^2 - 3)^3$$

$$\boxed{f'(x) = 40x(5x^2 - 3)^3}$$

13. $g(x) = \sqrt{4x+7} = (4x+7)^{1/2}$

$$g'(x) = \frac{1}{2}(4x+7)^{-1/2} \cdot 4 = \frac{4}{2\sqrt{4x+7}} = \frac{2}{\sqrt{4x+7}}$$

$$\boxed{g'(x) = \frac{2}{\sqrt{4x+7}}}$$

14. $h(x) = \sin(3x^2)$
Outer: $\sin(\,\cdot\,)$, inner: $3x^2$.

$$h'(x) = \cos(3x^2) \cdot 6x$$

$$\boxed{h'(x) = 6x\cos(3x^2)}$$

15. $f(x) = e^{2x^3 - x}$
Outer: $e^{(\,\cdot\,)}$, inner: $2x^3 - x$.

$$f'(x) = e^{2x^3 - x} \cdot (6x^2 - 1)$$

$$\boxed{f'(x) = (6x^2 - 1)e^{2x^3 - x}}$$

Part 5: Exponential and Logarithmic Derivatives

16. $f(x) = e^{5x}$
Chain rule with outer $e^{(\,\cdot\,)}$ and inner $5x$.

$$f'(x) = e^{5x} \cdot 5$$

$$\boxed{f'(x) = 5e^{5x}}$$

17. $g(x) = 3^x$
Recall: $\dfrac{d}{dx}[a^x] = a^x \ln a$.

$$g'(x) = 3^x \ln 3$$

$$\boxed{g'(x) = 3^x \ln 3}$$

18. $h(x) = \ln(x^2 + 1)$
Chain rule: $\dfrac{d}{dx}[\ln(u)] = \dfrac{u'}{u}$.

$$h'(x) = \frac{2x}{x^2 + 1}$$

$$\boxed{h'(x) = \frac{2x}{x^2 + 1}}$$

19. $f(x) = x^2 \cdot e^x$
Product rule. Let $u = x^2$, $v = e^x$.

$$u' = 2x \qquad v' = e^x$$

$$f'(x) = 2x \cdot e^x + x^2 \cdot e^x = e^x(2x + x^2)$$

$$\boxed{f'(x) = xe^x(x + 2)}$$

20. $g(x) = \ln(\sin(x))$
Chain rule: outer $\ln(\,\cdot\,)$, inner $\sin(x)$.

$$g'(x) = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

$$\boxed{g'(x) = \cot(x)}$$

Part 6: Trigonometric and Inverse Trigonometric Derivatives

21. $f(x) = \sin(x) + \cos(x)$

$$f'(x) = \cos(x) - \sin(x)$$

$$\boxed{f'(x) = \cos(x) - \sin(x)}$$

22. $g(x) = \tan(x) - \sec(x)$

$$g'(x) = \sec^2(x) - \sec(x)\tan(x)$$

$$\boxed{g'(x) = \sec^2(x) - \sec(x)\tan(x)}$$

23. $h(x) = x^2 \cos(x)$
Product rule. Let $u = x^2$, $v = \cos(x)$.

$$u' = 2x \qquad v' = -\sin(x)$$

$$h'(x) = 2x\cos(x) + x^2(-\sin(x)) = 2x\cos(x) - x^2\sin(x)$$
$$\boxed{h'(x) = 2x\cos(x) - x^2\sin(x)}$$

24. $f(x) = \arcsin(x)$
Recall the standard formula:

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$
$$\boxed{f'(x) = \frac{1}{\sqrt{1-x^2}}}$$

25. $g(x) = \arctan(3x)$
Recall: $\frac{d}{dx}[\arctan(u)] = \frac{u'}{1+u^2}$. Here $u = 3x$, so $u' = 3$.

$$g'(x) = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$$
$$\boxed{g'(x) = \frac{3}{1+9x^2}}$$

Part 7: Cotangent, Secant, and Cosecant

26. $f(x) = \cot(x)$

$$f'(x) = -\csc^2(x)$$
$$\boxed{f'(x) = -\csc^2(x)}$$

27. $g(x) = \sec(x) \cdot \tan(x)$
Product rule. Let $u = \sec(x)$, $v = \tan(x)$.

$$u' = \sec(x)\tan(x) \qquad v' = \sec^2(x)$$
$$g'(x) = \sec(x)\tan(x) \cdot \tan(x) + \sec(x) \cdot \sec^2(x)$$
$$= \sec(x)\tan^2(x) + \sec^3(x)$$
$$\boxed{g'(x) = \sec(x)\tan^2(x) + \sec^3(x)}$$

28. $h(x) = \csc(4x)$
Chain rule: outer $\csc(\,\cdot\,)$, inner $4x$.
Recall: $\frac{d}{dx}[\csc(u)] = -\csc(u)\cot(u) \cdot u'$.

$$h'(x) = -\csc(4x)\cot(4x) \cdot 4$$
$$\boxed{h'(x) = -4\csc(4x)\cot(4x)}$$

Part 8: Higher Order Derivatives

29. $f(x) = 5x^4 - 3x^2 + 2x$, find $f''(x)$.
Step 1: Find $f'(x)$.

$$f'(x) = 20x^3 - 6x + 2$$

Step 2: Differentiate again.

$$f''(x) = 60x^2 - 6$$

$$\boxed{f''(x) = 60x^2 - 6}$$

30. $f(x) = e^{2x}$, find $f'''(x)$.
Step 1: $f'(x) = 2e^{2x}$
Step 2: $f''(x) = 4e^{2x}$
Step 3: $f'''(x) = 8e^{2x}$

$$\boxed{f'''(x) = 8e^{2x}}$$

31. $f(x) = x\sin(x)$, find $f''(x)$.
Step 1: Find $f'(x)$ using the product rule. Let $u = x$, $v = \sin(x)$.

$$f'(x) = \sin(x) + x\cos(x)$$

Step 2: Differentiate again. Apply the product rule to $x\cos(x)$.

$$f''(x) = \cos(x) + \left[\cos(x) + x(-\sin(x))\right] = \cos(x) + \cos(x) - x\sin(x)$$

$$= 2\cos(x) - x\sin(x)$$

$$\boxed{f''(x) = 2\cos(x) - x\sin(x)}$$

Part 9: Implicit Differentiation

32. $x^2 + y^2 = 25$

Step 1: Differentiate both sides with respect to x. Remember y is a function of x, so use the chain rule on y^2.

$$2x + 2y\frac{dy}{dx} = 0$$

Step 2: Solve for $\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = -2x \implies \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

33. $x^3 + y^3 = 6xy$

Step 1: Differentiate both sides with respect to x. Use the product rule on $6xy$.

$$3x^2 + 3y^2\frac{dy}{dx} = 6\left(y + x\frac{dy}{dx}\right)$$

Step 2: Expand the right side.

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

Step 3: Collect all $\dfrac{dy}{dx}$ terms on the left.

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

Step 4: Factor out $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

Step 5: Divide.

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$\boxed{\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}}$$

34. $\sin(xy) = x$

Step 1: Differentiate both sides with respect to x. Use the chain rule on $\sin(xy)$. Inside xy, use the product rule.

$$\cos(xy) \cdot \frac{d}{dx}(xy) = 1$$

$$\cos(xy) \cdot \left(y + x\frac{dy}{dx}\right) = 1$$

Step 2: Expand.

$$y\cos(xy) + x\cos(xy)\frac{dy}{dx} = 1$$

Step 3: Isolate $\dfrac{dy}{dx}$.

$$x\cos(xy)\frac{dy}{dx} = 1 - y\cos(xy)$$

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy)}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy)}}$$

Part 10: Related Rates

35. Spherical balloon, $\frac{dr}{dt} = 2$ cm/s, find $\frac{dV}{dt}$ when $r = 5$ cm.

Step 1: Write the volume formula.

$$V = \frac{4}{3}\pi r^3$$

Step 2: Differentiate both sides with respect to time t.

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

Step 3: Substitute $r = 5$ and $\frac{dr}{dt} = 2$.

$$\frac{dV}{dt} = 4\pi(5)^2 \cdot 2 = 4\pi \cdot 25 \cdot 2 = 200\pi$$

$$\boxed{\frac{dV}{dt} = 200\pi \approx 628.3 \text{ cm}^3/\text{s}}$$

36. Ladder problem. Ladder length = 10 ft, $\frac{dx}{dt} = 1$ ft/s, find $\frac{dy}{dt}$ when $x = 6$ ft.

Step 1: Define variables. Let x = distance from wall to base, y = height on wall.

Step 2: Write the Pythagorean relationship.

$$x^2 + y^2 = 100$$

Step 3: Find y when $x = 6$.

$$36 + y^2 = 100 \implies y^2 = 64 \implies y = 8$$

Step 4: Differentiate both sides with respect to t.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Step 5: Substitute $x = 6$, $y = 8$, $\frac{dx}{dt} = 1$.

$$2(6)(1) + 2(8)\frac{dy}{dt} = 0 \implies 12 + 16\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-12}{16} = -\frac{3}{4}$$

$$\boxed{\frac{dy}{dt} = -\frac{3}{4} \text{ ft/s (the top is sliding down)}}$$

Part 11: Critical Points and Concavity

37. $f(x) = x^3 - 6x^2 + 9x + 1$

(a) Critical points: Set $f'(x) = 0$.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$$

$$f'(x) = 0 \implies x = 1 \text{ and } x = 3$$

(b) First derivative test:
Test sign of $f'(x)$ in the intervals $(-\infty, 1)$, $(1, 3)$, $(3, \infty)$.

$f'(0) = 3(0-1)(0-3) = 3(-1)(-3) = 9 > 0$ (increasing on $(-\infty, 1)$)
$f'(2) = 3(2-1)(2-3) = 3(1)(-1) = -3 < 0$ (decreasing on $(1, 3)$)
$f'(4) = 3(4-1)(4-3) = 3(3)(1) = 9 > 0$ (increasing on $(3, \infty)$)
$\Rightarrow x = 1$: increasing to decreasing $\Rightarrow$ **local maximum**, $f(1) = 5$.
$\Rightarrow x = 3$: decreasing to increasing $\Rightarrow$ **local minimum**, $f(3) = 1$.

(c) Concavity: Find $f''(x)$.

$$f''(x) = 6x - 12$$

Set $f''(x) = 0$: $6x - 12 = 0 \Rightarrow x = 2$.
$f''(x) < 0$ for $x < 2$ (concave down), $f''(x) > 0$ for $x > 2$ (concave up).

(d) Inflection point: At $x = 2$, concavity changes.

$$f(2) = 8 - 24 + 18 + 1 = 3$$

$$\boxed{\text{Inflection point at } (2,\ 3)}$$

38. $g(x) = x^4 - 8x^2$, second derivative test.
Step 1: Find $g'(x)$.

$$g'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$$

Critical points: $x = 0,\ x = 2,\ x = -2$.
Step 2: Find $g''(x)$.

$$g''(x) = 12x^2 - 16$$

Step 3: Evaluate g'' at each critical point.
$g''(0) = -16 < 0 \Rightarrow$ **local maximum** at $x = 0$, $g(0) = 0$.
$g''(2) = 48 - 16 = 32 > 0 \Rightarrow$ **local minimum** at $x = 2$, $g(2) = 16 - 32 = -16$.
$g''(-2) = 48 - 16 = 32 > 0 \Rightarrow$ **local minimum** at $x = -2$, $g(-2) = -16$.

$$\boxed{\text{Local max at } (0, 0);\ \text{local minima at } (2, -16) \text{ and } (-2, -16)}$$

Part 12: Optimization

39. Farmer's fence problem.

Step 1: Define variables. Let $x =$ the side parallel to the barn, $y =$ each of the two sides perpendicular to the barn.

Step 2: Write the constraint (fencing available).

$$x + 2y = 200$$

Step 3: Write the objective function (area to maximize).

$$A = xy$$

Step 4: Solve the constraint for x.

$$x = 200 - 2y$$

Step 5: Substitute into the area formula.

$$A(y) = (200 - 2y)y = 200y - 2y^2$$

Step 6: Find the critical point.

$$A'(y) = 200 - 4y = 0 \implies y = 50$$

Step 7: Find x.

$$x = 200 - 2(50) = 100$$

Step 8: Verify it is a maximum. $A''(y) = -4 < 0$ (concave down) $\Rightarrow$ maximum confirmed.

$$\boxed{\text{Dimensions: 100 m (parallel to barn)} \times \text{50 m, Area} = 5000\ \text{m}^2}$$

40. Cylindrical can. $V = 500\pi\ \text{cm}^3$.

Step 1: Write the volume constraint.

$$\pi r^2 h = 500\pi \implies h = \frac{500}{r^2}$$

Step 2: Write the surface area.

$$SA = 2\pi r^2 + 2\pi rh$$

Step 3: Substitute $h = \dfrac{500}{r^2}$.

$$SA(r) = 2\pi r^2 + 2\pi r \cdot \frac{500}{r^2} = 2\pi r^2 + \frac{1000\pi}{r}$$

Step 4: Differentiate and set equal to zero.

$$SA'(r) = 4\pi r - \frac{1000\pi}{r^2} = 0$$

$$4\pi r = \frac{1000\pi}{r^2} \implies 4r^3 = 1000 \implies r^3 = 250 \implies r = \sqrt[3]{250} \approx 6.30 \text{ cm}$$

Step 5: Find h.

$$h = \frac{500}{r^2} = \frac{500}{(\sqrt[3]{250})^2} = \frac{500}{250^{2/3}} \approx 12.60 \text{ cm}$$

Note: $h = 2r$, which is the classic result for an optimal cylinder.

$$\boxed{r = \sqrt[3]{250} \approx 6.30 \text{ cm}, \quad h \approx 12.60 \text{ cm}}$$

Part 13: Derivatives in Physics

41. $s(t) = t^3 - 6t^2 + 9t$

(a) Velocity:

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

$$\boxed{v(t) = 3t^2 - 12t + 9}$$

(b) Acceleration:

$$a(t) = v'(t) = 6t - 12$$

$$\boxed{a(t) = 6t - 12}$$

(c) Particle at rest: Set $v(t) = 0$.

$$3t^2 - 12t + 9 = 0 \implies 3(t^2 - 4t + 3) = 0 \implies 3(t-1)(t-3) = 0$$

$$\boxed{t = 1 \text{ s and } t = 3 \text{ s}}$$

(d) Speeding up or slowing down at $t = 4$?

$v(4) = 3(16) - 48 + 9 = 48 - 48 + 9 = 9 > 0$ (moving in positive direction)

$a(4) = 6(4) - 12 = 24 - 12 = 12 > 0$ (positive acceleration)

Since velocity and acceleration have the **same sign**, the particle is **speeding up** at $t = 4$.

Part 14: Linearization

42. $f(x) = \sqrt{x}$ at $a = 16$. Approximate $\sqrt{16.5}$.

Step 1: $f(16) = \sqrt{16} = 4$.

Step 2: $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$.

Step 3: Write $L(x)$.

$$L(x) = 4 + \frac{1}{8}(x - 16)$$

Step 4: Plug in $x = 16.5$.

$$L(16.5) = 4 + \frac{1}{8}(0.5) = 4 + 0.0625$$

$$\boxed{\sqrt{16.5} \approx 4.0625}$$

43. $f(x) = e^x$ at $a = 0$. Approximate $e^{-0.1}$.
Step 1: $f(0) = e^0 = 1$.
Step 2: $f'(x) = e^x$, so $f'(0) = 1$.
Step 3: $L(x) = 1 + 1 \cdot (x - 0) = 1 + x$.
Step 4: Plug in $x = -0.1$.

$$L(-0.1) = 1 + (-0.1) = 0.9$$

$$\boxed{e^{-0.1} \approx 0.9}$$

Part 15: Newton's Method

44. $f(x) = x^2 - 6$, $x_0 = 2$, three iterations.
Step 1: $f'(x) = 2x$.
Step 2: Iteration 1.

$$f(2) = 4 - 6 = -2, \quad f'(2) = 4$$

$$x_1 = 2 - \frac{-2}{4} = 2 + 0.5 = 2.5$$

Step 3: Iteration 2.

$$f(2.5) = 6.25 - 6 = 0.25, \quad f'(2.5) = 5$$

$$x_2 = 2.5 - \frac{0.25}{5} = 2.5 - 0.05 = 2.45$$

Step 4: Iteration 3.

$$f(2.45) = 6.0025 - 6 = 0.0025, \quad f'(2.45) = 4.9$$

$$x_3 = 2.45 - \frac{0.0025}{4.9} \approx 2.45 - 0.00051 \approx 2.44949$$

$$\boxed{\sqrt{6} \approx 2.4495}$$

45. $f(x) = x^3 + x - 5$, choose own x_0.
Step 1: Find $f'(x)$.

$$f'(x) = 3x^2 + 1$$

Step 2: Find a sign change.

$$f(1) = 1 + 1 - 5 = -3 \quad \text{(negative)}$$

$$f(2) = 8 + 2 - 5 = 5 \quad \text{(positive)}$$

Sign change between $x = 1$ and $x = 2$. Since $|f(1)| = 3$ and $|f(2)| = 5$, the root is closer to $x = 1$.

$$x_0 = 1$$

Step 3: Iteration 1.

$$f(1) = -3, \quad f'(1) = 3(1)^2 + 1 = 4$$

$$x_1 = 1 - \frac{-3}{4} = 1 + 0.75 = 1.75$$

Step 4: Iteration 2.

$$f(1.75) = (1.75)^3 + 1.75 - 5 = 5.35938 + 1.75 - 5 = 2.10938$$

$$f'(1.75) = 3(1.75)^2 + 1 = 3(3.0625) + 1 = 9.1875 + 1 = 10.1875$$

$$x_2 = 1.75 - \frac{2.10938}{10.1875} \approx 1.75 - 0.20707 \approx 1.54293$$

Step 5: Iteration 3.

$$f(1.54293) \approx (1.54293)^3 + 1.54293 - 5 \approx 3.67454 + 1.54293 - 5 \approx 0.21747$$

$$f'(1.54293) = 3(1.54293)^2 + 1 \approx 3(2.38064) + 1 \approx 7.14192 + 1 = 8.14192$$

$$x_3 \approx 1.54293 - \frac{0.21747}{8.14192} \approx 1.54293 - 0.02671 \approx 1.51622$$

$$\boxed{x \approx 1.516}$$

Thank you for learning with ApexMath!

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End of Answer Key • ApexMath
