Differential Equations — Comprehensive Review
ApexMath

This review covers all major topics from the differential equations unit. Work through each problem carefully, showing all steps. Use this worksheet to identify which topics you are confident in and which topics need more practice.

Topics Covered:

- Introduction to Differential Equations
- Slope Fields
- Separation of Variables
- Finding Particular Solutions
- Exponential Models
- Approximating Solutions Using Euler's Method
- Logistic Models

Part 1: Introduction to Differential Equations

1. Identify whether each equation is a differential equation. Write **Yes** or **No**.

(a) $\dfrac{dy}{dx} = 3x^2 + 1$

(b) $y = 5x - 7$

(c) $\dfrac{d^2y}{dx^2} + 4y = 0$

(d) $2x + 3 = 11$

2. Consider the differential equation:

$$\frac{dy}{dx} = 2x - 4$$

(a) Find the general solution by integrating both sides.

(b) Verify that $y = x^2 - 4x + C$ is a solution by differentiating it.

3. A differential equation is given as:

$$\frac{dy}{dx} = y$$

Show that $y = Ce^x$ is a solution for any constant C.

Part 2: Slope Fields

4. Consider the differential equation $\frac{dy}{dx} = x + y$.

Complete the table of slope values below.

x	y	$\frac{dy}{dx} = x + y$
0	0	
1	0	
0	1	
-1	2	
2	-1	

5. For the slope field of $\frac{dy}{dx} = -x$, answer the following:

(a) What is the slope at the point $(2, 5)$?

(b) What is the slope at the point $(-3, 1)$?

(c) Along what line(s) will all the slopes equal zero?

6. A slope field for a differential equation shows horizontal segments (slope $= 0$) along the line $y = 3$.

(a) What does this tell you about the differential equation along $y = 3$?

(b) Sketch a possible solution curve that passes through the point $(0, 5)$.

Part 3: Separation of Variables

7. Solve the following differential equation using separation of variables. Show every step.

$$\frac{dy}{dx} = 3y$$

8. Solve the following differential equation using separation of variables:

$$\frac{dy}{dx} = \frac{x}{y}$$

9. Solve the differential equation:

$$\frac{dy}{dx} = 2xy^2$$

10. Explain in your own words what it means to "separate variables." Why do we rearrange the equation before integrating?

Part 4: Finding Particular Solutions

11. Find the particular solution to the differential equation:

$$\frac{dy}{dx} = 4x^3$$

given the initial condition $y(1) = 6$.

12. Find the particular solution to the differential equation:

$$\frac{dy}{dx} = \frac{1}{x}, \quad x > 0$$

given that $y(1) = 0$.

13. Find the particular solution to the differential equation:

$$\frac{dy}{dx} = 2y$$

given that $y(0) = 5$.

Part 5: Exponential Models

14. A population of bacteria doubles every 3 hours. At time $t = 0$, there are 200 bacteria.

(a) Write a differential equation that models the population growth.

(b) Write the particular solution $P(t)$ using the exponential model $P(t) = P_0 e^{kt}$.

(c) How many bacteria are present after 9 hours?

15. A radioactive substance decays according to the equation:

$$\frac{dA}{dt} = -0.04A$$

At time $t = 0$, the amount is $A_0 = 500$ grams.

(a) Write the particular solution $A(t)$.

(b) How much of the substance remains after 10 years?

(c) Find the half-life of the substance (the time when $A = 250$). Leave your answer in exact form.

Part 6: Approximating Solutions Using Euler's Method

16. Use Euler's Method to approximate the solution to:

$$\frac{dy}{dx} = x + 1$$

with initial condition $y(0) = 2$, using step size $\Delta x = 1$.

Complete the table below. Carry out **3 steps**.

Step	x_n	y_n	$\left.\frac{dy}{dx}\right\|_{(x_n, y_n)}$
0	0	2	
1			
2			
3			

17. Use Euler's Method to approximate the value of $y(0.2)$ for the differential equation:

$$\frac{dy}{dx} = 2x - y$$

with initial condition $y(0) = 1$ and step size $\Delta x = 0.1$.

Show your work for each step clearly.

18. State whether Euler's Method gives an overestimate or underestimate in the following cases, and explain why.

(a) The solution curve is **concave up**.

(b) The solution curve is **concave down**.

Part 7: Logistic Models

19. The logistic differential equation is:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

(a) What does M represent in this model?

(b) What happens to $\dfrac{dP}{dt}$ as $P \to M$? Explain why this makes sense in context.

(c) At what value of P is the growth rate $\frac{dP}{dt}$ the greatest?

20. A population is modeled by the logistic equation with $k = 0.5$ and carrying capacity $M = 1000$. At time $t = 0$, the population is $P = 100$.

(a) Write out the specific differential equation for this population.

(b) Is the population growing or declining when $P = 100$? Justify using the equation.

(c) Is the population growing or declining when $P = 1200$? Justify.

(d) As $t \to \infty$, what value does P approach?

21. A logistic model has the solution:

$$P(t) = \frac{800}{1 + 7e^{-0.3t}}$$

(a) What is the carrying capacity?

(b) Find $P(0)$, the initial population.

(c) Find $P(10)$. Round to the nearest whole number.

Answer Key

Part 1: Introduction to Differential Equations

1.
(a) **Yes** — it contains a derivative $\frac{dy}{dx}$.
(b) **No** — it is a regular linear equation with no derivative.
(c) **Yes** — it contains a second derivative $\frac{d^2y}{dx^2}$.
(d) **No** — it is a simple algebraic equation.

2.
(a) Step 1: Integrate both sides with respect to x.

$$\int \frac{dy}{dx}\,dx = \int (2x - 4)\,dx$$

Step 2: Evaluate the integrals.

$$y = x^2 - 4x + C$$

$$\boxed{y = x^2 - 4x + C}$$

(b) Step 1: Differentiate $y = x^2 - 4x + C$ with respect to x.

$$\frac{dy}{dx} = 2x - 4$$

Step 2: This matches the original differential equation, so the solution is verified. ✓

3.
Step 1: Differentiate $y = Ce^x$ with respect to x.

$$\frac{dy}{dx} = Ce^x$$

Step 2: Notice that $Ce^x = y$ by substitution.

$$\frac{dy}{dx} = y$$

Step 3: This matches the original equation, confirming $y = Ce^x$ is a solution. ✓

Part 2: Slope Fields

4. Evaluate $\frac{dy}{dx} = x + y$ at each point.

x	y	$\frac{dy}{dx}$
0	0	0
1	0	1
0	1	1
-1	2	1
2	-1	1

5. The differential equation is $\frac{dy}{dx} = -x$.

(a) At $(2, 5)$: $\frac{dy}{dx} = -(2)$

$$\boxed{\frac{dy}{dx} = -2}$$

(b) At $(-3, 1)$: $\frac{dy}{dx} = -(-3)$

$$\boxed{\frac{dy}{dx} = 3}$$

(c) The slope equals zero when $-x = 0$, which means $x = 0$.

$$\boxed{x = 0 \text{ (the } y\text{-axis)}}$$

6.
(a) When $y = 3$, the rate of change is zero. This means the function is momentarily constant — the solution curve has a flat, horizontal tangent there.
(b) A solution curve through $(0, 5)$ would decrease toward $y = 3$ and then level off, approaching $y = 3$ as a horizontal asymptote.

Part 3: Separation of Variables

7.
Step 1: Separate y terms to the left and x terms (constants) to the right.

$$\frac{dy}{y} = 3\,dx$$

Step 2: Integrate both sides.

$$\int \frac{1}{y}\,dy = \int 3\,dx$$

$$\ln|y| = 3x + C$$

Step 3: Exponentiate both sides to solve for y.

$$|y| = e^{3x+C} = e^C \cdot e^{3x}$$

Step 4: Replace e^C with a new constant A (absorbing the $\pm$).

$$\boxed{y = Ae^{3x}}$$

8.

Step 1: Separate variables. Move y to the left and x to the right.

$$y\,dy = x\,dx$$

Step 2: Integrate both sides.

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

Step 3: Multiply both sides by 2.

$$y^2 = x^2 + C$$

$$\boxed{y^2 = x^2 + C}$$

9.

Step 1: Separate variables. Move all y terms left and x terms right.

$$\frac{dy}{y^2} = 2x\,dx$$

Step 2: Rewrite the left side.

$$y^{-2}\,dy = 2x\,dx$$

Step 3: Integrate both sides.

$$\int y^{-2}\,dy = \int 2x\,dx$$

$$-y^{-1} = x^2 + C$$

$$-\frac{1}{y} = x^2 + C$$

Step 4: Solve for y.

$$y = \frac{-1}{x^2 + C}$$

$$\boxed{y = \frac{-1}{x^2 + C}}$$

10. To separate variables means to rewrite the equation so that all y terms (including dy) are on one side and all x terms (including dx) are on the other. We do this so that we can integrate both sides independently — each side becomes a standard integral we already know how to solve.

Part 4: Finding Particular Solutions

11.

Step 1: Find the general solution by integrating.

$$y = \int 4x^3 \, dx = x^4 + C$$

Step 2: Apply the initial condition $y(1) = 6$.

$$6 = (1)^4 + C$$

$$6 = 1 + C$$

$$C = 5$$

Step 3: Write the particular solution.

$$\boxed{y = x^4 + 5}$$

12.

Step 1: Integrate both sides.

$$y = \int \frac{1}{x} \, dx = \ln|x| + C$$

Step 2: Apply the initial condition $y(1) = 0$.

$$0 = \ln(1) + C = 0 + C$$

$$C = 0$$

Step 3: Write the particular solution.

$$\boxed{y = \ln x}$$

13.
Step 1: Separate variables.

$$\frac{dy}{y} = 2\,dx$$

Step 2: Integrate both sides.

$$\ln|y| = 2x + C$$

Step 3: Solve for y.

$$y = Ae^{2x}$$

Step 4: Apply the initial condition $y(0) = 5$.

$$5 = Ae^{2(0)} = A \cdot 1 = A$$

$$A = 5$$

Step 5: Write the particular solution.

$$\boxed{y = 5e^{2x}}$$

Part 5: Exponential Models

14.
(a) Since the population grows proportionally to itself:

$$\boxed{\frac{dP}{dt} = kP}$$

(b) The general exponential model is $P(t) = P_0e^{kt}$, where $P_0 = 200$.
Step 1: Use the doubling condition. At $t = 3$, $P = 400$.

$$400 = 200e^{3k}$$

Step 2: Divide both sides by 200.

$$2 = e^{3k}$$

Step 3: Take the natural log of both sides.

$$\ln 2 = 3k \implies k = \frac{\ln 2}{3}$$

$$\boxed{P(t) = 200e^{(\ln 2)/3 \cdot t}}$$

(c) At $t = 9$:

$$P(9) = 200e^{(\ln 2/3)(9)} = 200e^{3\ln 2} = 200 \cdot 2^3 = 200 \cdot 8$$

$$\boxed{P(9) = 1600 \text{ bacteria}}$$

15.

(a) Step 1: Separate variables and integrate $\dfrac{dA}{dt} = -0.04A$.

$$A(t) = 500e^{-0.04t}$$

$$\boxed{A(t) = 500e^{-0.04t}}$$

(b) At $t = 10$:

$$A(10) = 500e^{-0.04(10)} = 500e^{-0.4}$$

$$A(10) \approx 500(0.6703) \approx 335.16$$

$$\boxed{A(10) \approx 335.16 \text{ grams}}$$

(c) Set $A(t) = 250$ and solve for t.

$$250 = 500e^{-0.04t}$$

$$\frac{1}{2} = e^{-0.04t}$$

$$\ln\left(\frac{1}{2}\right) = -0.04t$$

$$-\ln 2 = -0.04t$$

$$t = \frac{\ln 2}{0.04}$$

$$\boxed{t = \frac{\ln 2}{0.04} \approx 17.33 \text{ years}}$$

Part 6: Euler's Method

16.

Recall the Euler formula: $y_{n+1} = y_n + \frac{dy}{dx}\Big|_{(x_n, y_n)} \cdot \Delta x$

Step	x_n	y_n	$\frac{dy}{dx} = x_n + 1$
0	0	2	$0 + 1 = 1$
1	1	$2 + 1(1) = 3$	$1 + 1 = 2$
2	2	$3 + 2(1) = 5$	$2 + 1 = 3$
3	3	$5 + 3(1) = 8$	—

17.

Use $\frac{dy}{dx} = 2x - y$ with $y(0) = 1$ and $\Delta x = 0.1$.

Step 1: At $(x_0, y_0) = (0, 1)$:

$$\frac{dy}{dx} = 2(0) - 1 = -1$$

$$y_1 = 1 + (-1)(0.1) = 1 - 0.1 = 0.9$$

Step 2: At $(x_1, y_1) = (0.1,\ 0.9)$:

$$\frac{dy}{dx} = 2(0.1) - 0.9 = 0.2 - 0.9 = -0.7$$

$$y_2 = 0.9 + (-0.7)(0.1) = 0.9 - 0.07 = 0.83$$

$$\boxed{y(0.2) \approx 0.83}$$

18.

(a) When the solution is **concave up**, the curve bends upward. Euler's Method uses the slope at the left edge of each step, which is *below* the actual curve. This means Euler's Method gives an **underestimate**.

(b) When the solution is **concave down**, the curve bends downward. The slope at the left edge is *above* the actual curve. This means Euler's Method gives an **overestimate**.

Part 7: Logistic Models

19.
(a) M is the **carrying capacity** — the maximum population the environment can sustain.
(b) As $P \to M$, the factor $\left(1 - \frac{P}{M}\right) \to 0$. Therefore $\frac{dP}{dt} \to 0$. This makes sense because when the population reaches its maximum, there are no more resources for growth, so growth slows to zero.
(c) The growth rate is greatest when $P = \frac{M}{2}$.

$$\boxed{P = \frac{M}{2}}$$

20.
(a)

$$\boxed{\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{1000}\right)}$$

(b) At $P = 100$:

$$\frac{dP}{dt} = 0.5(100)\left(1 - \frac{100}{1000}\right) = 50(0.9) = 45 > 0$$

Since $\frac{dP}{dt} > 0$, the population is **growing**.
(c) At $P = 1200$:

$$\frac{dP}{dt} = 0.5(1200)\left(1 - \frac{1200}{1000}\right) = 600(1 - 1.2) = 600(-0.2) = -120 < 0$$

Since $\frac{dP}{dt} < 0$, the population is **declining** back toward the carrying capacity.
(d) As $t \to \infty$, $P \to M$.

$$\boxed{P \to 1000}$$

21.
(a) The carrying capacity is the value that P approaches as $t \to \infty$. As $t \to \infty$, $e^{-0.3t} \to 0$, so:

$$P \to \frac{800}{1 + 7(0)} = \frac{800}{1} = 800$$

$$\boxed{M = 800}$$

(b) At $t = 0$:

$$P(0) = \frac{800}{1 + 7e^{0}} = \frac{800}{1 + 7} = \frac{800}{8} = 100$$

$$\boxed{P(0) = 100}$$

(c) At $t = 10$:

$$P(10) = \frac{800}{1 + 7e^{-3}} = \frac{800}{1 + 7(0.04979)} = \frac{800}{1 + 0.3485} = \frac{800}{1.3485}$$

$$P(10) \approx 593$$

$$\boxed{P(10) \approx 593}$$

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