Exponential Models
ApexMath

Notes

The Big Idea: Quantities That Grow or Decay at a Constant Rate

Many real-world quantities change at a rate that is proportional to their current size. The more bacteria there are, the faster the colony grows. The more radioactive material remains, the faster it decays. This relationship is captured by a single differential equation:

$$\frac{dy}{dt} = ky$$

where y is the quantity at time t and k is the **proportionality constant**.

- If $k > 0$, the quantity **grows** over time (exponential growth).
- If $k < 0$, the quantity **decays** over time (exponential decay).

Solving the Model

This differential equation is separable. Let us solve it using separation of variables:
Step 1: Separate.

$$\frac{1}{y}\,dy = k\,dt$$

Step 2: Integrate both sides.

$$\ln|y| = kt + C$$

Step 3: Exponentiate and rename e^C as A:

$$y = Ae^{kt}$$

Step 4: Apply the initial condition $y(0) = y_0$.

$$y_0 = Ae^0 = A$$

The solution is:

$$\boxed{y(t) = y_0\, e^{kt}}$$

This is the **exponential model**. You will use this formula for every problem in this lesson.

Three Common Applications

1. Population and Bacterial Growth. A population starts at y_0 and grows at a continuous rate k. Use $y(t) = y_0e^{kt}$ where $k > 0$.

2. Radioactive Decay. A substance decays over time. The **half-life** $T_{1/2}$ is the time it takes for half the substance to remain. The constant k and the half-life are related by:

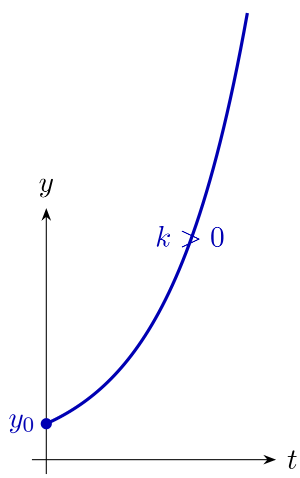
$$k = -\frac{\ln 2}{T_{1/2}}$$

This comes from solving $\frac{y_0}{2} = y_0 e^{k \cdot T_{1/2}}$, which gives $e^{kT_{1/2}} = \frac{1}{2}$, so $kT_{1/2} = -\ln 2$.

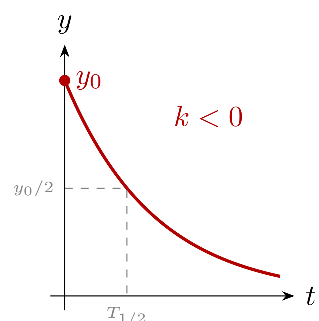
3. Newton's Law of Cooling. A hot (or cold) object placed in a room of constant temperature T_{room} cools (or warms) toward the room temperature. The temperature at time t is:

$$T(t) = T_{\text{room}} + (T_0 - T_{\text{room}})\, e^{kt}, \qquad k < 0$$

The key step is to let $u = T - T_{\text{room}}$. Then $\frac{du}{dt} = ku$ is the standard exponential model, and $u(t) = u_0 e^{kt}$.



Exponential Growth



Exponential Decay

Pro Tip

- **Always identify y_0, k, and what you are solving for before setting up any equation.** Writing down these three things first prevents most setup errors.
- **To find k, use a second data point.** You are given y_0 from the initial condition. A second value of y at a known time lets you solve for k by taking a natural log.
- **For half-life problems, use $k = -\ln 2/T_{1/2}$.** Do not re-derive this every time. Memorize the formula, then substitute the given half-life.
- **For Newton's Law of Cooling, work with $u = T - T_{\text{room}}$ first.** Solve for $u(t)$, then add T_{room} back at the end. This keeps the arithmetic cleaner.

- **When asked for a time, isolate the exponential and take** $\ln$ **of both sides.** The step $\ln(e^{kt}) = kt$ is the key to unlocking t from any exponential equation.

Examples

Example 1 (Bacterial Growth): A bacterial colony starts with 500 cells and grows to 1500 cells after 1 hour. Assume exponential growth.

(a) Find the growth constant k.
(b) Write the model $y(t)$.
(c) Find when the population reaches 6000 cells.

Part (a):
Step 1: Write the model with $y_0 = 500$.

$$y(t) = 500\, e^{kt}$$

Step 2: Use $y(1) = 1500$ to find k.

$$1500 = 500\, e^{k\cdot 1} \implies e^k = 3 \implies k = \ln 3$$

Part (b):

$$y(t) = 500\, e^{t\ln 3} = 500 \cdot 3^t$$

Part (c):

$$6000 = 500 \cdot 3^t \implies 3^t = 12 \implies t\ln 3 = \ln 12 \implies t = \frac{\ln 12}{\ln 3} \approx 2.26 \text{ hours}$$

$$\boxed{k = \ln 3, \quad y(t) = 500 \cdot 3^t, \quad t \approx 2.26 \text{ hrs}}$$

Example 2 (Radioactive Decay): Carbon-14 has a half-life of 5730 years. A sample initially contains 80 g. How much remains after 1000 years?

Step 1: Find k using the half-life formula.

$$k = -\frac{\ln 2}{5730}$$

Step 2: Write the model.

$$y(t) = 80\, e^{-t\ln 2/5730}$$

Step 3: Evaluate at $t = 1000$.

$$y(1000) = 80\, e^{-1000\ln 2/5730} = 80 \cdot 2^{-1000/5730}$$

$$\boxed{y(1000) \approx 70.89 \text{ g}}$$

Example 3 (Newton's Law of Cooling): A cup of coffee at 90°C is placed in a room at 20°C. After 5 minutes it has cooled to 60°C. Find the temperature after 10 minutes.

Step 1: Let $u = T - 20$. Then u satisfies $\dfrac{du}{dt} = ku$.

$$u(0) = 90 - 20 = 70, \qquad u(5) = 60 - 20 = 40$$

Step 2: Write the model and apply $u(5) = 40$.

$$u(t) = 70\, e^{kt}$$

$$40 = 70\, e^{5k} \implies e^{5k} = \frac{4}{7} \implies k = \frac{\ln(4/7)}{5}$$

Step 3: Find $u(10)$.

$$u(10) = 70\, e^{10k} = 70\left(e^{5k}\right)^2 = 70\left(\frac{4}{7}\right)^2 = 70 \cdot \frac{16}{49} = \frac{1120}{49} \approx 22.86$$

Step 4: Convert back to temperature.

$$T(10) = 20 + u(10) = 20 + \frac{1120}{49} = \frac{300}{7}$$

$$\boxed{T(10) = \frac{300}{7} \approx 42.86°\text{C}}$$

Example 4 (Doubling Time): An investment grows continuously at a rate of $k = 0.04$ (4% per year). How long does it take to double?

Step 1: Set $y(t) = 2y_0$ and solve for t.

$$2y_0 = y_0\, e^{0.04t} \implies e^{0.04t} = 2 \implies 0.04t = \ln 2$$

$$t = \frac{\ln 2}{0.04}$$

$$\boxed{t = \frac{\ln 2}{0.04} \approx 17.33 \text{ years}}$$

Common Mistakes

- **Using the wrong formula for half-life:** The half-life formula is $k = -\frac{\ln 2}{T_{1/2}}$. A common error is writing $k = \frac{1}{T_{1/2}}$ or forgetting the negative sign. Always derive or recall this carefully.
- **Forgetting to subtract the room temperature in cooling problems:** Newton's Law of Cooling applies to the *difference* $u = T - T_{\text{room}}$, not to T directly. If you write $T(t) = T_0 e^{kt}$ without subtracting T_{room}, your answer will be wrong.
- **Solving for k incorrectly by forgetting to take $\ln$:** To isolate k from $e^{kt} = r$, you must take the natural log of both sides: $kt = \ln r$. Forgetting this step and treating the exponent algebraically is a frequent error.
- **Confusing k with the growth rate percentage:** The equation $y = y_0 e^{kt}$ uses the *continuous* growth rate k. A stated rate of "5% per year" is sometimes a continuous rate ($k = 0.05$) and sometimes compound-period rate. Read the problem carefully.
- **Not checking units:** Make sure t and the given times are in the same units throughout the problem. Mixing hours and minutes, or years and days, will give nonsensical answers.

Worksheet

1. A bacteria colony starts with 200 cells and grows to 1600 cells after 3 hours. Assume exponential growth.

   (a) Find the growth constant k.

   (b) Write the model $y(t)$.

   (c) Find the population after 5 hours.

2. A radioactive substance has a half-life of 8 days. A sample starts with 120 g. How much remains after 24 days?

3. A radioactive substance decays according to $y(t) = y_0\, e^{-0.0231t}$ where t is in years. Find the half-life of this substance.

4. A metal rod is heated to 95°C and placed in a room at 25°C. After 4 minutes, its temperature is 65°C. Find the temperature after 8 minutes.

5. An investment of \$1000 grows continuously at an annual rate of $k = 0.05$. How many years until the investment reaches \$3000?

6. A city's population was 5000 in 2020 and 8000 in 2022. Assume exponential growth.
   (a) Find k.
   (b) Predict the population in 2026.

7. An archaeologist measures a Carbon-14 sample. The sample originally contained 50 g and currently contains 31.25 g. The half-life of Carbon-14 is 5730 years. How old is the sample?

8. (Challenge) A quantity y satisfies the exponential model. You are told that $y(1) = 6$ and $y(3) = 24$. Find $y_0 = y(0)$ and the constant k.

Answer Key

1. Bacteria: $y(0) = 200$, $y(3) = 1600$.

(a) Write the model: $y(t) = 200\, e^{kt}$.
Apply $y(3) = 1600$:

$$1600 = 200\, e^{3k} \implies e^{3k} = 8 \implies 3k = \ln 8 = 3\ln 2 \implies k = \ln 2$$

(b) Write the model:

$$y(t) = 200\, e^{t\ln 2} = 200 \cdot 2^t$$

(c) Find $y(5)$:

$$y(5) = 200 \cdot 2^5 = 200 \cdot 32$$

$$\boxed{k = \ln 2, \quad y(t) = 200 \cdot 2^t, \quad y(5) = 6400 \text{ cells}}$$

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2. Half-life 8 days, $y(0) = 120$ g. Find $y(24)$.

Step 1: Find k.

$$k = -\frac{\ln 2}{8}$$

Step 2: Write the model and evaluate at $t = 24$.
Note that 24 days is exactly 3 half-lives, so:

$$y(24) = 120 \cdot \left(\frac{1}{2}\right)^3 = 120 \cdot \frac{1}{8}$$

$$\boxed{y(24) = 15 \text{ g}}$$

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3. $y(t) = y_0\, e^{-0.0231t}$. Find the half-life.

Step 1: Set $y = \dfrac{y_0}{2}$ and solve for t.

$$\frac{y_0}{2} = y_0\, e^{-0.0231t} \implies \frac{1}{2} = e^{-0.0231t} \implies -0.0231t = \ln\frac{1}{2} = -\ln 2$$

$$t = \frac{\ln 2}{0.0231}$$

$$\boxed{T_{1/2} = \frac{\ln 2}{0.0231} \approx 30.0 \text{ years}}$$

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4. Cooling: $T_{\text{room}} = 25°\text{C}$, $T(0) = 95°\text{C}$, $T(4) = 65°\text{C}$. Find $T(8)$.

Step 1: Let $u = T - 25$.

$$u(0) = 95 - 25 = 70, \qquad u(4) = 65 - 25 = 40$$

Step 2: Write $u(t) = 70\,e^{kt}$ and apply $u(4) = 40$.

$$40 = 70\,e^{4k} \implies e^{4k} = \frac{4}{7} \implies k = \frac{\ln(4/7)}{4}$$

Step 3: Find $u(8)$.

$$u(8) = 70\,e^{8k} = 70\left(e^{4k}\right)^2 = 70\left(\frac{4}{7}\right)^2 = 70 \cdot \frac{16}{49} = \frac{1120}{49}$$

Step 4: Convert back.

$$T(8) = 25 + \frac{1120}{49} = \frac{1225 + 1120}{49} = \frac{2345}{49}$$

$$\boxed{T(8) = \frac{2345}{49} \approx 47.86°\text{C}}$$

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5. $y(0) = 1000$, $k = 0.05$. Find when $y = 3000$.

Step 1: Set up the equation.

$$3000 = 1000\,e^{0.05t} \implies e^{0.05t} = 3 \implies 0.05t = \ln 3$$

$$t = \frac{\ln 3}{0.05}$$

$$\boxed{t = \frac{\ln 3}{0.05} \approx 21.97 \text{ years}}$$

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6. Population: $y(0) = 5000$, $y(2) = 8000$.

(a) Write the model: $y(t) = 5000\,e^{kt}$.
Apply $y(2) = 8000$:

$$8000 = 5000\,e^{2k} \implies e^{2k} = \frac{8}{5} \implies k = \frac{\ln(8/5)}{2}$$

(b) $t = 6$ corresponds to the year 2026.

$$y(6) = 5000\,e^{6k} = 5000\left(e^{2k}\right)^3 = 5000\left(\frac{8}{5}\right)^3 = 5000 \cdot \frac{512}{125}$$

$$\boxed{k = \frac{\ln(8/5)}{2}, \quad y(2026) = 5000 \cdot \frac{512}{125} = 20{,}480 \text{ people}}$$

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7. Carbon-14: $y(0) = 50$ g, current $y = 31.25$ g, $T_{1/2} = 5730$ yrs.

Step 1: Write the model.

$$y(t) = 50\, e^{-t \ln 2/5730}$$

Step 2: Set $y = 31.25$ and solve for t.

$$31.25 = 50\, e^{-t \ln 2/5730} \implies e^{-t \ln 2/5730} = \frac{31.25}{50} = \frac{5}{8}$$

$$-\frac{t \ln 2}{5730} = \ln \frac{5}{8} = -\ln \frac{8}{5}$$

$$t = \frac{5730 \ln(8/5)}{\ln 2}$$

$$\boxed{t = \frac{5730 \ln(8/5)}{\ln 2} \approx 3885 \text{ years old}}$$

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8. (Challenge) $y(1) = 6$, $y(3) = 24$. Find y_0 and k.

Step 1: Write both equations.

$$y_0\, e^{k} = 6 \qquad \cdots (i)$$
$$y_0\, e^{3k} = 24 \qquad \cdots (ii)$$

Step 2: Divide equation (ii) by equation (i) to eliminate y_0.

$$\frac{y_0\, e^{3k}}{y_0\, e^{k}} = \frac{24}{6} \implies e^{2k} = 4 \implies 2k = \ln 4 = 2 \ln 2 \implies k = \ln 2$$

Step 3: Substitute $k = \ln 2$ into equation (i).

$$y_0\, e^{\ln 2} = 6 \implies 2y_0 = 6 \implies y_0 = 3$$

Step 4: Write the model.

$$y(t) = 3\, e^{t \ln 2} = 3 \cdot 2^t$$

Check: $y(1) = 3 \cdot 2 = 6$✓ $\quad y(3) = 3 \cdot 8 = 24$✓

$$\boxed{y_0 = 3, \quad k = \ln 2, \quad y(t) = 3 \cdot 2^t}$$

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