Finding Particular Solutions
ApexMath

Notes

What Is a Particular Solution?

In the previous lesson you learned how to separate variables and integrate to find a **general solution** — a family of curves that all satisfy the differential equation, distinguished by an arbitrary constant C.

A **particular solution** is the single curve from that family that passes through a specific point. You find it by using an **initial condition**: a given value $y(x_0) = y_0$ that pins down exactly which curve you want.

The process always follows the same two-stage structure:

Stage 1: Solve the DE — find the general solution $y = f(x, C)$

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Stage 2: Apply the initial condition — substitute x_0, y_0, solve for C

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Result: Particular solution — write the final answer with C replaced

Choosing the Correct Sign

When solving for y involves taking a square root, you get a $\pm$. The initial condition tells you which sign to keep.

For example, if $y^2 = x^2 + 4$ and $y(0) = -2$, then substituting $x = 0$:

$$y = \pm\sqrt{0 + 4} = \pm 2$$

Since $y(0) = -2$, keep the negative sign: $y = -\sqrt{x^2 + 4}$.

Implicit vs. Explicit Solutions

Sometimes solving for y explicitly is difficult or impossible. In that case it is perfectly acceptable to leave the solution in **implicit form**, as long as you have applied the initial condition to find C.

Explicit: $y = 3e^{x^2}$ (solved for y)

Implicit: $\dfrac{y^2}{2} + y = \dfrac{x^3}{3} + x + 4$ (not solved for y)

On the AP exam, if an explicit form is clean and possible, write it. If not, implicit form is accepted and full credit is given.

Pro Tip

- **Always find the general solution before applying the initial condition.** Do not substitute the initial condition partway through the algebra. Finish solving for y, then substitute.
- **After finding C, write the complete particular solution.** Do not stop at "$C = 5$." Replace C in the general solution and write the final answer.
- **Check your particular solution.** Substitute the initial condition back into your final answer and confirm it is satisfied. This catches most errors.
- **Use the initial condition to choose the $\pm$ sign.** If taking a square root produces a $\pm$, substitute the initial condition into both options and keep the one that matches.
- **Watch for** ln **steps.** When integrating $\frac{1}{y}$, you get $\ln|y|$. After exponentiating you get $y = Ae^{(\cdots)}$. Apply the initial condition to A, not to C — they are the same constant, just renamed.

Examples

Example 1: Solve $\frac{dy}{dx} = \frac{x^2}{y}$ with $y(0) = 3$.

Step 1: Separate.

$$y\,dy = x^2\,dx$$

Step 2: Integrate both sides.

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

Step 3: Apply $y(0) = 3$.

$$\frac{(3)^2}{2} = \frac{0}{3} + C \implies \frac{9}{2} = C$$

Step 4: Substitute $C = \frac{9}{2}$ and solve for y.

$$\frac{y^2}{2} = \frac{x^3}{3} + \frac{9}{2} \implies y^2 = \frac{2x^3}{3} + 9$$

Since $y(0) = 3 > 0$, take the positive square root:

$$\boxed{y = \sqrt{\frac{2x^3}{3} + 9}}$$

Example 2: Solve $\dfrac{dy}{dx} = -\dfrac{y}{x}$ with $y(2) = 6$.

Step 1: Separate.

$$\frac{1}{y}\,dy = -\frac{1}{x}\,dx$$

Step 2: Integrate both sides.

$$\ln|y| = -\ln|x| + C$$

Rewrite using $-\ln|x| = \ln\dfrac{1}{|x|}$, then exponentiate and rename e^C as A:

$$y = \frac{A}{x}$$

Step 3: Apply $y(2) = 6$.

$$6 = \frac{A}{2} \implies A = 12$$

$$\boxed{y = \frac{12}{x}}$$

Example 3: Solve $\dfrac{dy}{dx} = 2x(1+y)$ with $y(0) = 0$.

Step 1: Separate. Divide both sides by $(1+y)$:

$$\frac{1}{1+y}\,dy = 2x\,dx$$

Step 2: Integrate both sides.

$$\ln|1+y| = x^2 + C$$

Step 3: Exponentiate and rename e^C as A:

$$1 + y = Ae^{x^2} \implies y = Ae^{x^2} - 1$$

Step 4: Apply $y(0) = 0$.

$$0 = Ae^0 - 1 = A - 1 \implies A = 1$$

$$\boxed{y = e^{x^2} - 1}$$

Example 4: Solve $\dfrac{dy}{dx} = \dfrac{\cos x}{\sin y}$ with $y\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$.

Step 1: Separate.

$$\sin y\, dy = \cos x\, dx$$

Step 2: Integrate both sides.

$$-\cos y = \sin x + C$$

Step 3: Apply $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$.

$$-\cos\frac{\pi}{2} = \sin\frac{\pi}{2} + C \implies 0 = 1 + C \implies C = -1$$

Step 4: Substitute and solve for y.

$$-\cos y = \sin x - 1 \implies \cos y = 1 - \sin x \implies y = \arccos(1 - \sin x)$$

$$\boxed{y = \arccos(1 - \sin x)}$$

Example 5 (Implicit Solution): Solve $\dfrac{dy}{dx} = \dfrac{x^2 + 1}{y + 1}$ with $y(0) = 2$.

Step 1: Separate.

$$(y + 1)\, dy = (x^2 + 1)\, dx$$

Step 2: Integrate both sides.

$$\frac{y^2}{2} + y = \frac{x^3}{3} + x + C$$

Step 3: Apply $y(0) = 2$.

$$\frac{(2)^2}{2} + 2 = 0 + 0 + C \implies 2 + 2 = C \implies C = 4$$

Solving explicitly for y would require the quadratic formula, so we leave the answer in implicit form:

$$\boxed{\frac{y^2}{2} + y = \frac{x^3}{3} + x + 4}$$

Common Mistakes

- **Applying the initial condition too early:** Always complete the separation and integration first. The initial condition is only used at the very end to find C, after the general solution is written.

- **Forgetting to write the full particular solution after finding C:** Finding C is not the final step. You must substitute it back and write the complete equation for y.

- **Ignoring the $\pm$ sign when taking a square root:** If your general solution involves y^2, taking the square root gives $y = \pm\sqrt{\dots}$. Always use the initial condition to determine which sign applies.
- **Not checking the particular solution:** After finding the particular solution, substitute the initial condition back in to confirm it gives the correct value. This catches the vast majority of algebra errors.
- **Applying the initial condition to C instead of A after exponentiating:** When e^C is renamed A, apply the initial condition to A. If you write $y = Ae^{-\cos x}$ and then treat A as if it were still C, you may set up the equation incorrectly.

Worksheet

1. Solve $\dfrac{dy}{dx} = 6x$ with $y(1) = 5$.

2. Solve $\dfrac{dy}{dx} = y \sin x$ with $y(0) = 2$.

3. Solve $\dfrac{dy}{dx} = \dfrac{x}{y^2}$ with $y(0) = -2$.

4. Solve $\dfrac{dy}{dx} = \dfrac{e^{2x}}{y}$ with $y(0) = 1$.

5. Solve $\dfrac{dy}{dx} = \dfrac{x+1}{y-1}$ with $y(0) = 3$.

6. Solve $\dfrac{dy}{dx} = 4xy^2$ with $y(1) = 1$.

7. Solve $\dfrac{dy}{dx} = \sqrt{y} \cdot e^x$ with $y(0) = 4$.

8. (Challenge) Solve $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + 1}$ with $y(0) = 5$.

Answer Key

1. $\dfrac{dy}{dx} = 6x, \quad y(1) = 5.$

Step 1: Separate and integrate.

$$dy = 6x\,dx \implies y = 3x^2 + C$$

Step 2: Apply $y(1) = 5$.

$$5 = 3(1)^2 + C = 3 + C \implies C = 2$$

$$\boxed{y = 3x^2 + 2}$$

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2. $\dfrac{dy}{dx} = y \sin x, \quad y(0) = 2.$

Step 1: Separate.

$$\frac{1}{y}\,dy = \sin x\,dx$$

Step 2: Integrate both sides.

$$\ln|y| = -\cos x + C$$

Step 3: Exponentiate and rename e^C as A:

$$y = Ae^{-\cos x}$$

Step 4: Apply $y(0) = 2$.

$$2 = Ae^{-\cos 0} = Ae^{-1} \implies A = 2e$$

Step 5: Write the particular solution.

$$y = 2e \cdot e^{-\cos x} = 2e^{1-\cos x}$$

$$\boxed{y = 2e^{1-\cos x}}$$

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3. $\dfrac{dy}{dx} = \dfrac{x}{y^2}, \quad y(0) = -2.$

Step 1: Separate.

$$y^2\,dy = x\,dx$$

Step 2: Integrate both sides.

$$\frac{y^3}{3} = \frac{x^2}{2} + C$$

Step 3: Apply $y(0) = -2$.

$$\frac{(-2)^3}{3} = 0 + C \implies -\frac{8}{3} = C$$

Step 4: Substitute and solve for y. Multiply through by 3:

$$y^3 = \frac{3x^2}{2} - 8$$

$$\boxed{y = \sqrt[3]{\frac{3x^2}{2} - 8}}$$

Check: $y(0) = \sqrt[3]{-8} = -2$. ✓

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4. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{y}, \quad y(0) = 1.$

Step 1: Separate.

$$y\,dy = e^{2x}\,dx$$

Step 2: Integrate both sides.

$$\frac{y^2}{2} = \frac{e^{2x}}{2} + C \implies y^2 = e^{2x} + 2C$$

Rename $2C$ as C:

$$y^2 = e^{2x} + C$$

Step 3: Apply $y(0) = 1$.

$$1 = e^0 + C = 1 + C \implies C = 0$$

$$y^2 = e^{2x}$$

Step 4: Since $y(0) = 1 > 0$, take the positive root:

$$y = \sqrt{e^{2x}} = e^x$$

$$\boxed{y = e^x}$$

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5. $\dfrac{dy}{dx} = \dfrac{x+1}{y-1}, \quad y(0) = 3.$

Step 1: Separate.

$$(y-1)\,dy = (x+1)\,dx$$

Step 2: Integrate both sides.

$$\frac{(y-1)^2}{2} = \frac{(x+1)^2}{2} + C$$

$$(y-1)^2 = (x+1)^2 + 2C$$

Rename $2C$ as C:

$$(y-1)^2 = (x+1)^2 + C$$

Step 3: Apply $y(0) = 3$.

$$(3-1)^2 = (0+1)^2 + C \implies 4 = 1 + C \implies C = 3$$

Step 4: Solve for y. Since $y(0) = 3 > 1$, take the positive root:

$$y - 1 = \sqrt{(x+1)^2 + 3}$$

$$\boxed{y = 1 + \sqrt{(x+1)^2 + 3}}$$

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6. $\dfrac{dy}{dx} = 4xy^2, \quad y(1) = 1.$

Step 1: Separate. Divide both sides by y^2:

$$\frac{1}{y^2}\,dy = 4x\,dx \implies y^{-2}\,dy = 4x\,dx$$

Step 2: Integrate both sides.

$$-\frac{1}{y} = 2x^2 + C$$

Step 3: Apply $y(1) = 1$.

$$-\frac{1}{1} = 2(1)^2 + C \implies -1 = 2 + C \implies C = -3$$

$$-\frac{1}{y} = 2x^2 - 3$$

Step 4: Solve for y.

$$y = \frac{-1}{2x^2-3} = \frac{1}{3-2x^2}$$

$$\boxed{y = \frac{1}{3-2x^2}}$$

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7. $\dfrac{dy}{dx} = \sqrt{y} \cdot e^x, \quad y(0) = 4.$

Step 1: Separate. Write $\sqrt{y} = y^{1/2}$ and divide:

$$y^{-1/2}\, dy = e^x\, dx$$

Step 2: Integrate both sides.

$$\int y^{-1/2}\, dy = \int e^x\, dx \implies 2\sqrt{y} = e^x + C$$

Step 3: Apply $y(0) = 4$.

$$2\sqrt{4} = e^0 + C \implies 4 = 1 + C \implies C = 3$$

$$2\sqrt{y} = e^x + 3$$

Step 4: Solve for y. Divide by 2 and square:

$$\sqrt{y} = \frac{e^x + 3}{2} \implies y = \left(\frac{e^x + 3}{2}\right)^2$$

$$\boxed{y = \left(\frac{e^x + 3}{2}\right)^2}$$

Check: $y(0) = \left(\dfrac{1+3}{2}\right)^2 = 4.$ ✓

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8. (Challenge) $\dfrac{dy}{dx} = \dfrac{xy}{x^2+1}, \quad y(0) = 5.$

Step 1: Separate.

$$\frac{1}{y}\, dy = \frac{x}{x^2+1}\, dx$$

Step 2: Integrate both sides. For the right side, use the substitution $u = x^2 + 1$, $du = 2x\, dx$, so $x\, dx = \dfrac{du}{2}$:

$$\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1)$$

Therefore:

$$\ln|y| = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C$$

Step 3: Exponentiate and rename e^C as A:

$$y = A\sqrt{x^2+1}$$

Step 4: Apply $y(0) = 5$.

$$5 = A\sqrt{0+1} = A \implies A = 5$$

$$\boxed{y = 5\sqrt{x^2+1}}$$

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