The Fundamental Theorem of Calculus
ApexMath

Notes

What is the Fundamental Theorem of Calculus?

The Fundamental Theorem of Calculus (FTC) is the most important result in all of calculus. It reveals that **differentiation and integration are inverse operations** — they undo each other. The theorem has two parts, and each one is a powerful tool on its own.

FTC Part 1: Differentiating an Integral

If f is continuous on $[a, b]$, then the function defined by:

$$g(x) = \int_a^x f(t)\,dt$$

is differentiable, and its derivative is simply $f(x)$:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

In plain words: if you integrate a function and then differentiate the result, you get the original function back. The lower limit a is just a constant and plays no role in the derivative. The variable t inside the integral is a dummy variable — it disappears when you differentiate.

FTC Part 1 with the Chain Rule

If the upper limit is not just x but a function of x, say $g(x)$, then you must apply the Chain Rule:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$$

Think of it this way: first plug the upper limit into f (replacing t with $g(x)$), then multiply by the derivative of the upper limit.

FTC Part 2: Evaluating a Definite Integral

If f is continuous on $[a, b]$ and F is any antiderivative of f (meaning $F'(x) = f(x)$), then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This is the result that makes integration practical. Instead of computing a limit of Riemann sums every time, we can find an antiderivative and simply evaluate it at the two endpoints.

We use the shorthand notation:

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

The bracket $\Big[F(x)\Big]_a^b$ means: evaluate $F(x)$ at $x = b$, then subtract $F(x)$ evaluated at $x = a$.

Quick Reference: Common Antiderivatives

You need to know these to apply FTC Part 2. A full antiderivatives lesson follows, but here are the essentials:

Function $f(x)$	**Antiderivative** $F(x)$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\sec^2(x)$	$\tan(x)$
$\csc^2(x)$	$-\cot(x)$
$\sec(x)\tan(x)$	$\sec(x)$

Note: when using FTC Part 2, you do **not** write $+C$. The constant cancels out when you subtract $F(a)$ from $F(b)$, so it is always safe to drop it.

The Big Picture: How the Two Parts Connect

FTC Part 1 says: $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ (differentiation undoes integration)

FTC Part 2 says: $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ (integration undoes differentiation)

Together, they confirm that the two core operations of calculus are two sides of the same coin.

Pro Tip

- For FTC Part 1, the lower limit does not matter — only the upper limit determines the answer. Students sometimes waste time trying to do something with the lower limit. Ignore it; just plug the upper limit into f.

- When the upper limit involves a Chain Rule, write out the two steps clearly: (1) replace t with the upper limit expression in f, (2) multiply by the derivative of that upper limit. Do not try to do both in one step.
- For FTC Part 2, find the antiderivative first and write it inside the bracket $\Big[F(x)\Big]_a^b$ before plugging in any numbers. Plugging in while also finding the antiderivative at the same time leads to errors.
- Always subtract in the correct order: $F(b) - F(a)$, *not* $F(a) - F(b)$. The upper limit goes first.
- For FTC Part 2, do not add $+C$. It cancels and is unnecessary. Writing it and then forgetting to cancel it causes sign errors.

Examples

Example 1 (FTC Part 1 — Basic): Find $\dfrac{d}{dx}\displaystyle\int_2^x (t^3 + \sin t)\, dt$.

Step 1: Identify the integrand.
The function inside the integral is $f(t) = t^3 + \sin t$.

Step 2: Apply FTC Part 1.
Replace t with x in the integrand. The lower limit 2 is a constant and disappears.

$$\frac{d}{dx}\int_2^x (t^3 + \sin t)\, dt = x^3 + \sin x$$

$$\boxed{\frac{d}{dx}\int_2^x (t^3 + \sin t)\, dt = x^3 + \sin x}$$

Example 2 (FTC Part 1 — Chain Rule): Find $\dfrac{d}{dx}\displaystyle\int_0^{x^2} e^t\, dt$.

Step 1: Identify the integrand and the upper limit.
The integrand is $f(t) = e^t$. The upper limit is $g(x) = x^2$, so $g'(x) = 2x$.

Step 2: Apply FTC Part 1 with the Chain Rule.

$$\frac{d}{dx}\int_0^{g(x)} f(t)\, dt = f(g(x)) \cdot g'(x)$$

Step 2a: Replace t with $g(x) = x^2$ in $f(t) = e^t$:

$$f(g(x)) = e^{x^2}$$

Step 2b: Multiply by $g'(x) = 2x$:

$$\frac{d}{dx} \int_0^{x^2} e^t \, dt = e^{x^2} \cdot 2x$$

$$\boxed{\frac{d}{dx} \int_0^{x^2} e^t \, dt = 2xe^{x^2}}$$

Example 3 (FTC Part 2 — Polynomial): Evaluate $\displaystyle\int_1^3 (3x^2 - 4x + 1) \, dx$.

Step 1: Find the antiderivative of $f(x) = 3x^2 - 4x + 1$.
Apply the reverse power rule to each term:

$$F(x) = x^3 - 2x^2 + x$$

Step 2: Write the evaluation bracket.

$$\int_1^3 (3x^2 - 4x + 1) \, dx = \Big[x^3 - 2x^2 + x\Big]_1^3$$

Step 3: Evaluate at $x = 3$.

$$F(3) = (3)^3 - 2(3)^2 + 3 = 27 - 18 + 3 = 12$$

Step 4: Evaluate at $x = 1$.

$$F(1) = (1)^3 - 2(1)^2 + 1 = 1 - 2 + 1 = 0$$

Step 5: Subtract.

$$F(3) - F(1) = 12 - 0 = 12$$

$$\boxed{\int_1^3 (3x^2 - 4x + 1) \, dx = 12}$$

Example 4 (FTC Part 2 — Trig and Exponential): Evaluate $\displaystyle\int_0^{\pi/2} \cos(x) \, dx$.

Step 1: Find the antiderivative.
The antiderivative of $\cos(x)$ is $\sin(x)$.

$$F(x) = \sin(x)$$

Step 2: Write the evaluation bracket.

$$\int_0^{\pi/2} \cos(x)\,dx = \Big[\sin(x)\Big]_0^{\pi/2}$$

Step 3: Evaluate at $x = \pi/2$ and $x = 0$.

$$F\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \qquad F(0) = \sin(0) = 0$$

Step 4: Subtract.

$$1 - 0 = 1$$

$$\boxed{\int_0^{\pi/2} \cos(x)\,dx = 1}$$

Example 5 (FTC Part 2 — Exponential): Evaluate $\displaystyle\int_0^2 e^x\,dx$.

Step 1: Find the antiderivative.
The antiderivative of e^x is e^x.

$$F(x) = e^x$$

Step 2: Write the evaluation bracket and compute.

$$\int_0^2 e^x\,dx = \Big[e^x\Big]_0^2 = e^2 - e^0 = e^2 - 1$$

$$\boxed{\int_0^2 e^x\,dx = e^2 - 1}$$

Common Mistakes

- **Differentiating the lower limit in FTC Part 1:** The lower limit is always a constant. Only the upper limit matters. Do not differentiate or substitute the lower limit.
- **Forgetting the Chain Rule in FTC Part 1:** If the upper limit is x^2, $\sin(x)$, or any function of x, you must multiply by its derivative. The answer is never just $f(\text{upper limit})$ alone when the chain rule applies.
- **Subtracting in the wrong order:** It is always $F(b) - F(a)$. Reversing this flips the sign of your entire answer.
- **Arithmetic errors when evaluating $F(a)$:** Students are careful with $F(b)$ but rush through $F(a)$. Take the same care with both. A sign error on $F(a)$ ruins the whole answer.

- **Writing $+C$ in a definite integral:** The $+C$ is only for indefinite integrals. In FTC Part 2, you are evaluating at specific numbers, so $+C$ always cancels and should not be written.
- **Using the wrong antiderivative:** Always double-check by differentiating $F(x)$. If $F'(x) = f(x)$, the antiderivative is correct.

Worksheet

Part A: FTC Part 1

1. Find $\dfrac{d}{dx}\displaystyle\int_3^x (5t^2 - 2)\,dt$.

2. Find $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{t^4+1}\,dt$.

3. Find $\dfrac{d}{dx}\displaystyle\int_x^5 \cos(t^2)\,dt$.

   Hint: Reverse the limits first. Recall that $\displaystyle\int_x^5 = -\int_5^x$.

4. Find $\dfrac{d}{dx}\displaystyle\int_1^{x^3} \ln(t)\,dt$.

5. Find $\dfrac{d}{dx}\displaystyle\int_0^{\sin(x)} t^2\,dt$.

6. Find $\dfrac{d}{dx}\displaystyle\int_x^{x^2} e^t\,dt$.

Hint: Split into two integrals using a constant c: $\displaystyle\int_x^{x^2} = \int_c^{x^2} - \int_c^{x}$.

Part B: FTC Part 2

7. $\displaystyle\int_0^4 (2x+3)\,dx$

8. $\displaystyle\int_{-1}^2 (x^3-3x)\,dx$

9. $\displaystyle\int_1^4 \sqrt{x}\,dx$

10. $\displaystyle\int_0^\pi \sin(x)\,dx$

11. $\displaystyle\int_1^e \frac{1}{x}\,dx$

12. $\displaystyle\int_0^1 (e^x + x^2)\,dx$

13. (Challenge) $\displaystyle\int_0^{\pi/4} \sec^2(x)\,dx$

14. (Challenge) Find $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \sin(t^2)\, dt$.

Answer Key

Part A: FTC Part 1

1. $\dfrac{d}{dx}\displaystyle\int_3^x (5t^2 - 2)\,dt$

Apply FTC Part 1 directly. Replace t with x in the integrand. The lower limit 3 is irrelevant.

$$\boxed{\frac{d}{dx}\int_3^x (5t^2 - 2)\,dt = 5x^2 - 2}$$

2. $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{t^4+1}\,dt$

Apply FTC Part 1 directly. Replace t with x.

$$\boxed{\frac{d}{dx}\int_0^x \sqrt{t^4+1}\,dt = \sqrt{x^4+1}}$$

3. $\dfrac{d}{dx}\displaystyle\int_x^5 \cos(t^2)\,dt$

Step 1: Reverse the limits. Reversing swaps the sign:

$$\int_x^5 \cos(t^2)\,dt = -\int_5^x \cos(t^2)\,dt$$

Step 2: Apply FTC Part 1 to $\displaystyle\int_5^x \cos(t^2)\,dt$ and carry the negative sign:

$$\frac{d}{dx}\left[-\int_5^x \cos(t^2)\,dt\right] = -\cos(x^2)$$

$$\boxed{\frac{d}{dx}\int_x^5 \cos(t^2)\,dt = -\cos(x^2)}$$

4. $\dfrac{d}{dx}\displaystyle\int_1^{x^3} \ln(t)\,dt$

The upper limit is $g(x) = x^3$, so $g'(x) = 3x^2$. Apply FTC Part 1 with Chain Rule.
Step 1: Replace t with x^3 in $f(t) = \ln(t)$:

$$f(g(x)) = \ln(x^3)$$

Step 2: Multiply by $g'(x) = 3x^2$:

$$\frac{d}{dx}\int_1^{x^3} \ln(t)\,dt = \ln(x^3)\cdot 3x^2$$

Simplify using $\ln(x^3) = 3\ln(x)$:

$$= 3\ln(x) \cdot 3x^2 = 9x^2\ln(x)$$

$$\boxed{\frac{d}{dx}\int_1^{x^3} \ln(t)\,dt = 9x^2\ln(x)}$$

5. $\dfrac{d}{dx}\displaystyle\int_0^{\sin(x)} t^2\,dt$

The upper limit is $g(x) = \sin(x)$, so $g'(x) = \cos(x)$.

Step 1: Replace t with $\sin(x)$ in $f(t) = t^2$:

$$f(g(x)) = (\sin(x))^2 = \sin^2(x)$$

Step 2: Multiply by $g'(x) = \cos(x)$:

$$\frac{d}{dx}\int_0^{\sin(x)} t^2\,dt = \sin^2(x) \cdot \cos(x)$$

$$\boxed{\frac{d}{dx}\int_0^{\sin(x)} t^2\,dt = \sin^2(x)\cos(x)}$$

6. $\dfrac{d}{dx}\displaystyle\int_x^{x^2} e^t\,dt$

Step 1: Split using a constant c:

$$\int_x^{x^2} e^t\,dt = \int_c^{x^2} e^t\,dt - \int_c^{x} e^t\,dt$$

Step 2: Differentiate each part separately using FTC Part 1 with Chain Rule.

For $\displaystyle\int_c^{x^2} e^t\,dt$: upper limit $g(x) = x^2$, $g'(x) = 2x$:

$$\frac{d}{dx}\int_c^{x^2} e^t\,dt = e^{x^2} \cdot 2x$$

For $\displaystyle\int_c^{x} e^t\,dt$: upper limit is just x, no chain rule needed:

$$\frac{d}{dx}\int_c^{x} e^t\,dt = e^x$$

Step 3: Subtract.

$$\frac{d}{dx}\int_x^{x^2} e^t\,dt = 2xe^{x^2} - e^x$$

$$\boxed{\frac{d}{dx}\int_x^{x^2} e^t\,dt = 2xe^{x^2} - e^x}$$

Part B: FTC Part 2

7. $\displaystyle\int_0^4 (2x+3)\,dx$

Step 1: Find the antiderivative.

$$F(x) = x^2 + 3x$$

Step 2: Evaluate.

$$\Big[x^2 + 3x\Big]_0^4 = \big[(4)^2 + 3(4)\big] - \big[(0)^2 + 3(0)\big] = [16 + 12] - [0] = 28$$

$$\boxed{\int_0^4 (2x+3)\,dx = 28}$$

8. $\displaystyle\int_{-1}^2 (x^3 - 3x)\,dx$

Step 1: Find the antiderivative.

$$F(x) = \frac{x^4}{4} - \frac{3x^2}{2}$$

Step 2: Evaluate at $x = 2$:

$$F(2) = \frac{16}{4} - \frac{3(4)}{2} = 4 - 6 = -2$$

Step 3: Evaluate at $x = -1$:

$$F(-1) = \frac{1}{4} - \frac{3(1)}{2} = \frac{1}{4} - \frac{6}{4} = -\frac{5}{4}$$

Step 4: Subtract.

$$F(2) - F(-1) = -2 - \left(-\frac{5}{4}\right) = -2 + \frac{5}{4} = -\frac{8}{4} + \frac{5}{4}$$

$$\boxed{\int_{-1}^2 (x^3 - 3x)\,dx = -\frac{3}{4}}$$

9. $\displaystyle\int_1^4 \sqrt{x}\,dx$

Step 1: Rewrite as $x^{1/2}$ and find the antiderivative.

$$F(x) = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}$$

Step 2: Evaluate.

$$\left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2} = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{16}{3} - \frac{2}{3}$$

$$\boxed{\int_1^4 \sqrt{x}\,dx = \frac{14}{3}}$$

10. $\displaystyle\int_0^\pi \sin(x)\,dx$

Step 1: Antiderivative of $\sin(x)$ is $-\cos(x)$.
Step 2: Evaluate.

$$\Big[-\cos(x)\Big]_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) + \cos(0) = 1 + 1$$

$$\boxed{\int_0^\pi \sin(x)\,dx = 2}$$

11. $\displaystyle\int_1^e \frac{1}{x}\,dx$

Step 1: Antiderivative of $\dfrac{1}{x}$ is $\ln|x|$.
Step 2: Evaluate.

$$\Big[\ln|x|\Big]_1^e = \ln(e) - \ln(1) = 1 - 0$$

$$\boxed{\int_1^e \frac{1}{x}\,dx = 1}$$

12. $\displaystyle\int_0^1 (e^x + x^2)\,dx$

Step 1: Find the antiderivative term by term.

$$F(x) = e^x + \frac{x^3}{3}$$

Step 2: Evaluate at $x = 1$:

$$F(1) = e^1 + \frac{1}{3} = e + \frac{1}{3}$$

Step 3: Evaluate at $x = 0$:

$$F(0) = e^0 + \frac{0}{3} = 1$$

Step 4: Subtract.

$$F(1) - F(0) = e + \frac{1}{3} - 1$$

$$\boxed{\int_0^1 (e^x + x^2)\,dx = e - \frac{2}{3}}$$

13. (Challenge) $\int_0^{\pi/4} \sec^2(x)\,dx$
Step 1: Antiderivative of $\sec^2(x)$ is $\tan(x)$.
Step 2: Evaluate.

$$\Big[\tan(x)\Big]_0^{\pi/4} = \tan\Big(\frac{\pi}{4}\Big) - \tan(0) = 1 - 0$$

$$\boxed{\int_0^{\pi/4} \sec^2(x)\,dx = 1}$$

14. (Challenge) $\frac{d}{dx}\int_{x^2}^{x^3} \sin(t^2)\,dt$
Step 1: Split using a constant c:

$$\int_{x^2}^{x^3} \sin(t^2)\,dt = \int_c^{x^3} \sin(t^2)\,dt - \int_c^{x^2} \sin(t^2)\,dt$$

Step 2: Differentiate the first part. Upper limit $g(x) = x^3$, $g'(x) = 3x^2$:

$$\frac{d}{dx}\int_c^{x^3} \sin(t^2)\,dt = \sin\big((x^3)^2\big)\cdot 3x^2 = 3x^2\sin(x^6)$$

Step 3: Differentiate the second part. Upper limit $g(x) = x^2$, $g'(x) = 2x$:

$$\frac{d}{dx}\int_c^{x^2} \sin(t^2)\,dt = \sin\big((x^2)^2\big)\cdot 2x = 2x\sin(x^4)$$

Step 4: Subtract.

$$\frac{d}{dx}\int_{x^2}^{x^3} \sin(t^2)\,dt = 3x^2\sin(x^6) - 2x\sin(x^4)$$

$$\boxed{\frac{d}{dx}\int_{x^2}^{x^3} \sin(t^2)\,dt = 3x^2\sin(x^6) - 2x\sin(x^4)}$$

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