Lesson 9: Implicit Differentiation

ApexMath

Notes

- Some functions are not solved explicitly as $y = f(x)$, e.g. $x^2 + y^2 = 25$.
- **Implicit Differentiation:** Differentiate both sides of the equation with respect to x, treating y as a function of x.
- When differentiating y, multiply by dy/dx (chain rule):

$$\frac{d}{dx}[y^n] = ny^{n-1}\frac{dy}{dx}$$

- Solve for $\frac{dy}{dx}$ after differentiating.

Pro Tips

- Always differentiate **term by term**.
- Don't forget the $\frac{dy}{dx}$ factor when differentiating y.
- After differentiating, collect all terms with $\frac{dy}{dx}$ on one side.
- Check your algebra carefully when solving for $\frac{dy}{dx}$.

Examples

Example 1: Differentiate $x^2 + y^2 = 25$.
Step 1 – Differentiate both sides:

$$2x + 2y\frac{dy}{dx} = 0$$

Step 2 – Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Example 2: Differentiate $xy + y^2 = 10$.
Step 1 – Differentiate both sides:

$$(x\frac{dy}{dx} + y) + 2y\frac{dy}{dx} = 0$$

Step 2 – Combine like terms:

$$(x + 2y)\frac{dy}{dx} + y = 0$$

Step 3 – Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = -\frac{y}{x + 2y}}$$

Example 3: Differentiate $\sin(xy) = x + y$.

Step 1 – Differentiate both sides:

$$\cos(xy)(x\frac{dy}{dx} + y) = 1 + \frac{dy}{dx}$$

Step 2 – Collect $\frac{dy}{dx}$ terms:

$$x\cos(xy)\frac{dy}{dx} - \frac{dy}{dx} = 1 - y\cos(xy)$$

Step 3 – Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) - 1}}$$

Worksheet

Differentiate each implicitly and solve for $\frac{dy}{dx}$:

1. $x^2 + y^2 = 49$
2. $x^2y + y^3 = 7$
3. $e^y + xy = x^2$
4. $\ln(xy) = x + y$
5. $x^3 + y^3 = 6xy$
6. **Bonus:** Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 25$

Answer Key with Steps

1. $x^2 + y^2 = 49$ **Step 1 – Differentiate both sides:**

$$2x + 2y\frac{dy}{dx} = 0$$

   Step 2 – Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

$x^2y + y^3 = 7$ **Step 1 – Differentiate both sides:**

$$2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

Step 2 – Factor $\frac{dy}{dx}$:

$$(x^2 + 3y^2)\frac{dy}{dx} = -2xy$$

Step 3 – Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = -\frac{2xy}{x^2 + 3y^2}}$$

$e^y + xy = x^2$ **Step 1 – Differentiate both sides:**

$$e^y\frac{dy}{dx} + x\frac{dy}{dx} + y = 2x$$

Step 2 – Factor $\frac{dy}{dx}$:

$$(e^y + x)\frac{dy}{dx} = 2x - y$$

Step 3 – Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{2x - y}{e^y + x}}$$

$\ln(xy) = x + y$ **Step 1 – Differentiate both sides:**

$$\frac{1}{xy}(x\frac{dy}{dx} + y) = 1 + \frac{dy}{dx}$$

Step 2 – Multiply through by xy:

$$x\frac{dy}{dx} + y = xy + xy\frac{dy}{dx}$$

Step 3 – Collect $\frac{dy}{dx}$ terms and solve:

$$x\frac{dy}{dx} - xy\frac{dy}{dx} = xy - y \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y(x-1)}{x(1-y)}$$

$$\boxed{\frac{dy}{dx} = \frac{y(x-1)}{x(1-y)}}$$

$x^3 + y^3 = 6xy$ **Step 1 – Differentiate both sides:**

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

Step 2 – Collect $\frac{dy}{dx}$ terms:

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

Step 3 – Solve for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

Bonus: $x^2 + y^2 = 25$ (find $\frac{d^2y}{dx^2}$) **Step 1 – First derivative:**

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Step 2 – Differentiate again:

$$2 + 2\left(\frac{dy}{dx} \cdot \frac{dy}{dx} + y\frac{d^2y}{dx^2}\right) = 0$$

Step 3 – Substitute $\frac{dy}{dx} = -\frac{x}{y}$ and solve:

$$2 + 2\left(\frac{x^2}{y^2} + y\frac{d^2y}{dx^2}\right) = 0 \quad \Rightarrow y\frac{d^2y}{dx^2} = -1 - \frac{x^2}{y^2}$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}}$$

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