Improper Integrals
ApexMath

Notes

What Makes an Integral Improper?

A definite integral $\int_a^b f(x)\,dx$ is called **improper** when one of two things happens:

- **Type 1 — Infinite limits:** One or both of the limits of integration is ∞ or $-\infty$.
- **Type 2 — Discontinuous integrand:** The function $f(x)$ has a vertical asymptote somewhere on or inside the interval $[a, b]$.

In both cases, the standard FTC formula cannot be applied directly. Instead, we replace the problematic limit or point with a variable t and take a limit.

Type 1: Infinite Limits of Integration

When the upper limit is ∞, replace it with t and take the limit as $t \to \infty$:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

When the lower limit is $-\infty$, replace it with t and take the limit as $t \to -\infty$:

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

When **both** limits are infinite, you must split the integral at any convenient finite value c (usually $c = 0$) and handle each piece separately:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

Both pieces must be evaluated as limits independently.

Type 2: Discontinuous Integrand

If $f(x)$ has a vertical asymptote at $x = b$ (the upper limit), replace b with t and take a one-sided limit from the left:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

If the asymptote is at $x = a$ (the lower limit), replace a with t and take a one-sided limit from the right:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

If the asymptote is at an interior point $x = c$ where $a < c < b$, split the integral at c and handle each piece with its own limit:

$$\int_a^b f(x)\,dx = \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

Important: You must split at every point of discontinuity. If you try to evaluate straight through an asymptote without splitting, you will get a wrong answer.

Summary Table

Situation	**What to do**
Upper limit $= \infty$	$\lim_{t \to \infty} \int_a^t f\,dx$
Lower limit $= -\infty$	$\lim_{t \to -\infty} \int_t^b f\,dx$
Both limits infinite	Split at c: $\int_{-\infty}^c + \int_c^\infty$
Asymptote at upper limit b	$\lim_{t \to b^-} \int_a^t f\,dx$
Asymptote at lower limit a	$\lim_{t \to a^+} \int_t^b f\,dx$
Asymptote inside at c	Split at c: $\lim_{t \to c^-} \int_a^t + \lim_{t \to c^+} \int_t^b$

Key Limits to Remember:

$$\lim_{t \to \infty} e^{-t} = 0 \qquad \lim_{t \to \infty} \frac{1}{t^n} = 0 \text{ for } n > 0 \qquad \lim_{t \to \infty} \ln(t) = \infty$$

$$\lim_{t \to 0^+} \ln(t) = -\infty \qquad \lim_{t \to 0^+} \frac{1}{t^n} = \infty \text{ for } n > 0$$

Pro Tip

- Always write the limit notation before integrating. Setting up $\lim_{t \to \infty} \int_a^t$ first keeps the work organized and reminds you to evaluate the limit at the end.
- After integrating, substitute t in place of the original bound, then evaluate the limit. Do not substitute ∞ directly into the antiderivative — always go through the limit process.
- When splitting $\int_{-\infty}^{\infty}$, you can split at any finite value. Splitting at $c = 0$ is almost always the cleanest choice.

- For Type 2, look at the integrand carefully before writing the limits. Ask: does the denominator equal zero anywhere on $[a, b]$? Does the function blow up? Identify every bad point first.
- When you have an asymptote inside the interval, you must use *two separate* limit variables — one approaching c from the left and one from the right. Using a single limit variable for both sides is incorrect.

Examples

Example 1 (Type 1 — Upper Limit ∞): Evaluate $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$.

Step 1: Replace ∞ with t and write the limit.

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t x^{-2}\,dx$$

Step 2: Find the antiderivative and evaluate at the limits.

$$= \lim_{t\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{1}\right)$$

Step 3: Evaluate the limit. As $t \to \infty$, $\dfrac{1}{t} \to 0$.

$$= \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1 - 0 = 1$$

$$\boxed{\int_1^{\infty} \frac{1}{x^2}\,dx = 1}$$

Example 2 (Type 1 — Lower Limit $-\infty$): Evaluate $\displaystyle\int_{-\infty}^{0} e^x\,dx$.

Step 1: Replace $-\infty$ with t and write the limit.

$$\int_{-\infty}^{0} e^x\,dx = \lim_{t\to-\infty} \int_t^0 e^x\,dx$$

Step 2: Integrate and evaluate.

$$= \lim_{t\to-\infty} \left[e^x\right]_t^0 = \lim_{t\to-\infty} \left(e^0 - e^t\right) = \lim_{t\to-\infty} (1 - e^t)$$

Step 3: Evaluate the limit. As $t \to -\infty$, $e^t \to 0$.

$$= 1 - 0 = 1$$

$$\boxed{\int_{-\infty}^{0} e^x \, dx = 1}$$

Example 3 (Type 1 — Both Limits Infinite): Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$.

Step 1: Split at $c = 0$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx + \int_{0}^{\infty} \frac{1}{1+x^2} \, dx$$

Step 2: Evaluate each piece using limits. Recall $\displaystyle\int \frac{1}{1+x^2} \, dx = \arctan(x) + C$.
Left piece:

$$\lim_{t \to -\infty} \Big[\arctan(x) \Big]_t^0 = \lim_{t \to -\infty} (\arctan(0) - \arctan(t)) = 0 - \left(-\frac{\pi}{2} \right) = \frac{\pi}{2}$$

Right piece:

$$\lim_{t \to \infty} \Big[\arctan(x) \Big]_0^t = \lim_{t \to \infty} (\arctan(t) - \arctan(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Step 3: Add the two pieces.

$$\frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \pi}$$

Example 4 (Type 2 — Asymptote at Lower Limit): Evaluate $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{x}} \, dx$.

Step 1: Identify the discontinuity. $f(x) = x^{-1/2}$ blows up at $x = 0$, which is the lower limit.
Step 2: Replace the lower limit with t and take the limit from the right.

$$\int_{0}^{1} \frac{1}{\sqrt{x}} \, dx = \lim_{t \to 0^+} \int_{t}^{1} x^{-1/2} \, dx$$

Step 3: Integrate.

$$= \lim_{t \to 0^+} \left[\frac{x^{1/2}}{1/2} \right]_t^1 = \lim_{t \to 0^+} \Big[2\sqrt{x} \Big]_t^1 = \lim_{t \to 0^+} \left(2\sqrt{1} - 2\sqrt{t} \right) = \lim_{t \to 0^+} (2 - 2\sqrt{t})$$

Step 4: Evaluate the limit. As $t \to 0^+$, $\sqrt{t} \to 0$.

$$= 2 - 0 = 2$$

$$\boxed{\int_0^1 \frac{1}{\sqrt{x}}\, dx = 2}$$

Example 5 (Type 2 — Asymptote Inside the Interval): Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$.

Step 1: Identify the discontinuity. $f(x) = \dfrac{1}{x^2}$ has a vertical asymptote at $x = 0$, which lies *inside* $[-1, 1]$.

Step 2: Split at $x = 0$ and handle each piece with its own limit.

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x^2}\, dx + \lim_{s \to 0^+} \int_{s}^{1} \frac{1}{x^2}\, dx$$

Step 3: Evaluate the left piece.

$$\lim_{t \to 0^-} \left[-\frac{1}{x}\right]_{-1}^{t} = \lim_{t \to 0^-} \left(-\frac{1}{t} + \frac{1}{-1}\right) = \lim_{t \to 0^-} \left(-\frac{1}{t} - 1\right)$$

As $t \to 0^-$, $-\dfrac{1}{t} \to +\infty$. So the left piece diverges to $+\infty$.

Step 4: Since one piece diverges, the entire integral diverges.

$$\boxed{\int_{-1}^{1} \frac{1}{x^2}\, dx \text{ diverges}}$$

Warning: A common mistake is to ignore the asymptote at $x = 0$ and compute $\left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2$. This is completely wrong because it treats the integrand as if it were continuous, which it is not. Always check for asymptotes before evaluating.

Common Mistakes

- **Substituting ∞ directly into the antiderivative:** You cannot write $F(\infty)$. Always replace ∞ with a variable t first, evaluate $F(t)$, then take $\lim_{t \to \infty} F(t)$.
- **Ignoring a vertical asymptote inside the interval:** If the integrand blows up at an interior point and you do not split the integral there, you will compute a nonsense answer. Always scan the integrand for asymptotes across the entire interval before starting.

- **Using a single limit variable for a split integral:** When splitting at an interior discontinuity, use two *independent* variables t (approaching from the left) and s (approaching from the right). Using the same variable for both sides is mathematically incorrect.
- **Forgetting to take the one-sided limit for Type 2:** At a lower limit asymptote, the limit must be from the right ($t \to a^+$). At an upper limit asymptote, it must be from the left ($t \to b^-$). Using the wrong direction produces a limit over a region where the function is undefined.
- **Adding the pieces before evaluating each limit:** Evaluate each integral piece to a number (or determine it diverges) *before* adding. If even one piece diverges, the whole integral diverges — you cannot add finite and infinite pieces together.

Worksheet

Part A: Type 1 — Infinite Limits

1. Evaluate $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$.

2. Evaluate $\displaystyle\int_0^\infty e^{-2x}\,dx$.

3. Evaluate $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^2}\,dx$.

4. Evaluate $\int_1^\infty \frac{1}{x}\,dx$.

5. Evaluate $\int_{-\infty}^\infty xe^{-x^2}\,dx$.

(Split at 0. Use u-substitution on each piece.)

Part B: Type 2 — Discontinuous Integrand

6. Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$.

(Asymptote at the upper limit $x = 4$.)

7. Evaluate $\int_0^1 \frac{1}{x^{2/3}}\,dx$.

(Asymptote at the lower limit $x = 0$.)

8. Evaluate $\displaystyle\int_0^3 \frac{1}{(x-1)^{1/3}}\,dx$.

(Asymptote at interior point $x = 1$. Split the integral there.)

9. (Challenge) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\,dx$.

(Hint: $\frac{1}{e^x+e^{-x}} = \frac{e^x}{e^{2x}+1}$. Let $u = e^x$ and split at 0.)

10. (Challenge) Evaluate $\displaystyle\int_0^{\infty} xe^{-x}\,dx$.

(Use integration by parts, then evaluate the resulting limit carefully.)

Answer Key

1. $\int_1^\infty \frac{1}{x^3}\,dx$

$$= \lim_{t\to\infty} \int_1^t x^{-3}\,dx = \lim_{t\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2}\right) = 0+\frac{1}{2}$$

$$\boxed{\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{2}}$$

2. $\int_0^\infty e^{-2x}\,dx$

$$= \lim_{t\to\infty} \int_0^t e^{-2x}\,dx = \lim_{t\to\infty} \left[-\frac{e^{-2x}}{2}\right]_0^t = \lim_{t\to\infty}\left(-\frac{e^{-2t}}{2}+\frac{1}{2}\right) = 0+\frac{1}{2}$$

$$\boxed{\int_0^\infty e^{-2x}\,dx = \frac{1}{2}}$$

3. $\int_{-\infty}^{-1} \frac{1}{x^2}\,dx$

$$= \lim_{t\to-\infty} \int_t^{-1} x^{-2}\,dx = \lim_{t\to-\infty} \left[-\frac{1}{x}\right]_t^{-1} = \lim_{t\to-\infty}\left(-\frac{1}{-1}+\frac{1}{t}\right) = \lim_{t\to-\infty}\left(1+\frac{1}{t}\right) = 1+0$$

$$\boxed{\int_{-\infty}^{-1} \frac{1}{x^2}\,dx = 1}$$

4. $\int_1^\infty \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \Big[\ln|x|\Big]_1^t = \lim_{t\to\infty}(\ln t - \ln 1) = \lim_{t\to\infty} \ln t = \infty$$

$$\boxed{\int_1^\infty \frac{1}{x}\,dx \text{ diverges}}$$

5. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

Split at 0:

$$= \lim_{t\to-\infty} \int_t^0 xe^{-x^2}\,dx + \lim_{s\to\infty}\int_0^s xe^{-x^2}\,dx$$

For each piece, use $u = -x^2$, $du = -2x\,dx$, so $x\,dx = -\dfrac{du}{2}$:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + C$$

Left piece:

$$\lim_{t\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-t^2}\right) = -\frac{1}{2} + 0 = -\frac{1}{2}$$

Right piece:

$$\lim_{s\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^s = \lim_{s\to\infty}\left(-\frac{1}{2}e^{-s^2} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}$$

$$-\frac{1}{2} + \frac{1}{2} = 0$$

$$\boxed{\int_{-\infty}^{\infty} xe^{-x^2}\,dx = 0}$$

(This makes sense geometrically: xe^{-x^2} is an odd function, so its integral over a symmetric interval is zero.)

6. $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$
Asymptote at $x = 4$ (upper limit). Replace upper limit with $t \to 4^-$.

$$= \lim_{t\to4^-}\int_0^t (4-x)^{-1/2}\,dx$$

Use $u = 4 - x$, $du = -dx$:

$$\int (4-x)^{-1/2}\,dx = -2\sqrt{4-x}$$

$$= \lim_{t\to4^-}\left[-2\sqrt{4-x}\right]_0^t = \lim_{t\to4^-}\left(-2\sqrt{4-t} + 2\sqrt{4}\right) = -2(0) + 4 = 4$$

$$\boxed{\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = 4}$$

7. $\displaystyle\int_0^1 \frac{1}{x^{2/3}}\,dx$
Asymptote at $x = 0$ (lower limit). Replace with $t \to 0^+$.

$$= \lim_{t\to0^+}\int_t^1 x^{-2/3}\,dx = \lim_{t\to0^+}\left[\frac{x^{1/3}}{1/3}\right]_t^1 = \lim_{t\to0^+}\left[3x^{1/3}\right]_t^1$$

$$= \lim_{t \to 0^+} (3 - 3t^{1/3}) = 3 - 0 = 3$$

$$\boxed{\int_0^1 \frac{1}{x^{2/3}}\, dx = 3}$$

8. $\displaystyle\int_0^3 \frac{1}{(x-1)^{1/3}}\, dx$
Asymptote at interior point $x = 1$. Split there and use two independent limit variables.

$$= \lim_{t \to 1^-} \int_0^t (x-1)^{-1/3}\, dx + \lim_{s \to 1^+} \int_s^3 (x-1)^{-1/3}\, dx$$

Antiderivative: $\displaystyle\int (x-1)^{-1/3}\, dx = \frac{(x-1)^{2/3}}{2/3} = \frac{3}{2}(x-1)^{2/3}$
Left piece:

$$\lim_{t \to 1^-} \left[\frac{3}{2}(x-1)^{2/3}\right]_0^t = \lim_{t \to 1^-} \left(\frac{3}{2}(t-1)^{2/3} - \frac{3}{2}(-1)^{2/3}\right)$$

Note: $(-1)^{2/3} = ((-1)^2)^{1/3} = 1^{1/3} = 1$.

$$= \frac{3}{2}(0) - \frac{3}{2}(1) = -\frac{3}{2}$$

Right piece:

$$\lim_{s \to 1^+} \left[\frac{3}{2}(x-1)^{2/3}\right]_s^3 = \lim_{s \to 1^+} \left(\frac{3}{2}(2)^{2/3} - \frac{3}{2}(s-1)^{2/3}\right) = \frac{3}{2} \cdot 2^{2/3} - 0 = \frac{3}{2} \cdot \sqrt[3]{4}$$

Add the pieces:

$$-\frac{3}{2} + \frac{3\sqrt[3]{4}}{2} = \frac{3}{2}\left(\sqrt[3]{4} - 1\right)$$

$$\boxed{\int_0^3 \frac{1}{(x-1)^{1/3}}\, dx = \frac{3}{2}\left(\sqrt[3]{4} - 1\right)}$$

9. (Challenge) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx$
Rewrite: $\displaystyle\frac{1}{e^x + e^{-x}} = \frac{e^x}{e^{2x} + 1}$.
Split at 0:

$$= \lim_{t \to -\infty} \int_t^0 \frac{e^x}{e^{2x} + 1}\, dx + \lim_{s \to \infty} \int_0^s \frac{e^x}{e^{2x} + 1}\, dx$$

For each piece, let $u = e^x$, $du = e^x\, dx$:

$$\int \frac{e^x}{e^{2x} + 1}\, dx = \int \frac{du}{u^2 + 1} = \arctan(u) = \arctan(e^x) + C$$

Left piece: limits $u = e^t \to 0$ as $t \to -\infty$, to $u = e^0 = 1$.

$$\lim_{t\to-\infty} \Big[\arctan(e^x)\Big]_t^0 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Right piece: limits $u = 1$ to $u = e^s \to \infty$ as $s \to \infty$.

$$\lim_{s\to\infty} \Big[\arctan(e^x)\Big]_0^s = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\,dx = \frac{\pi}{2}}$$

10. (Challenge) $\displaystyle\int_0^{\infty} xe^{-x}\,dx$

$$= \lim_{t\to\infty} \int_0^t xe^{-x}\,dx$$

Apply IBP: $u = x$, $dv = e^{-x}\,dx$, $du = dx$, $v = -e^{-x}$.

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1)$$

Evaluate:

$$\lim_{t\to\infty} \Big[-e^{-x}(x+1)\Big]_0^t = \lim_{t\to\infty} \big(-e^{-t}(t+1) + e^0(0+1)\big) = \lim_{t\to\infty} \left(-\frac{t+1}{e^t} + 1\right)$$

As $t \to \infty$, $\dfrac{t+1}{e^t} \to 0$ (exponential grows faster than polynomial):

$$= 0 + 1 = 1$$

$$\boxed{\int_0^{\infty} xe^{-x}\,dx = 1}$$

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