Indefinite Integrals
ApexMath

Notes

What is an Indefinite Integral?

In the previous lesson we found antiderivatives by reversing differentiation. An **indefinite integral** is simply the formal notation for doing exactly that. When we write:

$$\int f(x)\,dx = F(x) + C$$

we are saying: find all functions whose derivative is $f(x)$. Each part of this notation has a specific meaning:

- $\int$ is the **integral sign** — it tells you to find the antiderivative.
- $f(x)$ is the **integrand** — the function you are integrating.
- dx tells you the **variable of integration** — in this case, x. It also signals where the integrand ends.
- $F(x) + C$ is the **general antiderivative** — the complete family of all antiderivatives.

Indefinite vs. Definite Integrals

It is important not to confuse these two types of integrals. They look similar but produce very different things:

	Indefinite Integral	**Definite Integral**
Notation	$\int f(x)\,dx$	$\int_a^b f(x)\,dx$
Result	A function $F(x) + C$	A number
Limits	None	Lower limit a, upper limit b
Includes $+C$?	Yes, always	No

The definite integral uses the indefinite integral as a tool: once you find $F(x)$ from the indefinite integral, you evaluate $F(b) - F(a)$ to get the number.

The Linearity Properties

These two properties let you break complicated integrals into simpler pieces:

Constant Multiple Rule:

$$\int k \cdot f(x)\, dx = k \int f(x)\, dx$$

A constant factor can be pulled out in front of the integral sign.

Sum and Difference Rule:

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

An integral of a sum or difference can be split into separate integrals.

Together these are called **linearity**. They mean you can integrate term by term and handle constants freely. This is the same reason derivatives could be taken term by term.

Important: What You Cannot Do

Linearity works for sums and constant multiples. It does **not** work for products or quotients:

$$\int f(x) \cdot g(x)\, dx \neq \int f(x)\, dx \cdot \int g(x)\, dx$$

$$\int \frac{f(x)}{g(x)}\, dx \neq \frac{\int f(x)\, dx}{\int g(x)\, dx}$$

Products and quotients require special techniques covered in later lessons. For now, the strategy is to **rewrite** products and quotients algebraically into a sum before integrating.

Splitting a Fraction

One very useful rewriting technique is splitting a rational expression by dividing each term in the numerator by the denominator separately:

$$\int \frac{f(x) + g(x)}{h(x)}\, dx = \int \frac{f(x)}{h(x)}\, dx + \int \frac{g(x)}{h(x)}\, dx$$

This only works when the *denominator* is a single term. For example:

$$\frac{x^3 + 5x}{x} = \frac{x^3}{x} + \frac{5x}{x} = x^2 + 5$$

After splitting, each piece becomes a simple power of x that can be integrated directly.

Connecting Back to FTC

The indefinite integral and the definite integral are linked by the Fundamental Theorem of Calculus. The workflow is:

Step 1: Find the indefinite integral $\displaystyle\int f(x)\, dx = F(x) + C$.
Step 2: Drop the $+C$ (it cancels in the subtraction).
Step 3: Evaluate $F(b) - F(a)$.

The indefinite integral is not just abstract notation — it is the engine that powers every definite integral calculation.

Pro Tip

- Always write dx at the end of every integral. It is not decorative — it tells you which variable you are integrating with respect to. In later lessons with substitution, this will become critical.
- When you see a fraction inside an integral, always check if the denominator is a single term. If it is, split the fraction immediately. This converts a potentially difficult integral into simple power rule problems.
- Pull constants out front *before* integrating. It keeps the expression cleaner and reduces the chance of arithmetic errors inside the integral.
- Never split a fraction when the denominator is a *sum*. For example, $\dfrac{x^2}{x+1}$ cannot be split this way. Only split when the denominator is a single monomial like x, x^2, or $\sqrt{x}$.
- After finishing any indefinite integral, differentiate your answer to check it. If the derivative matches the original integrand, the answer is correct.

Examples

Example 1 (Linearity — Constant Multiple and Sum): Find $\displaystyle\int (10x^4 - 6x^2 + 14)\,dx$.

Step 1: Apply the sum rule to split the integral.

$$= \int 10x^4\,dx - \int 6x^2\,dx + \int 14\,dx$$

Step 2: Apply the constant multiple rule to pull constants out.

$$= 10\int x^4\,dx - 6\int x^2\,dx + 14\int dx$$

Step 3: Integrate each piece using the reverse power rule.

$$= 10 \cdot \frac{x^5}{5} - 6 \cdot \frac{x^3}{3} + 14x + C$$

Step 4: Simplify.

$$= 2x^5 - 2x^3 + 14x + C$$

$$\boxed{\int (10x^4 - 6x^2 + 14)\,dx = 2x^5 - 2x^3 + 14x + C}$$

Example 2 (Splitting a Fraction): Find $\displaystyle\int \frac{3x^4 - 8x^2 + 2}{x^2}\,dx$.

Step 1: The denominator is a single term x^2, so split the fraction.

$$\frac{3x^4 - 8x^2 + 2}{x^2} = \frac{3x^4}{x^2} - \frac{8x^2}{x^2} + \frac{2}{x^2} = 3x^2 - 8 + 2x^{-2}$$

Step 2: Integrate term by term.

$$\int \left(3x^2 - 8 + 2x^{-2}\right) dx = 3 \cdot \frac{x^3}{3} - 8x + 2 \cdot \frac{x^{-1}}{-1} + C$$

Step 3: Simplify.

$$= x^3 - 8x - \frac{2}{x} + C$$

$$\boxed{\int \frac{3x^4 - 8x^2 + 2}{x^2}\, dx = x^3 - 8x - \frac{2}{x} + C}$$

Example 3 (Mixed Functions): Find $\displaystyle\int \left(\frac{5}{x} - 3\sin(x) + 4e^x\right) dx$.

Step 1: Split by the sum rule and pull out constants.

$$= 5\int \frac{1}{x}\, dx - 3\int \sin(x)\, dx + 4\int e^x\, dx$$

Step 2: Apply the antiderivative rules to each term.

$$\int \frac{1}{x}\, dx = \ln|x|, \qquad \int \sin(x)\, dx = -\cos(x), \qquad \int e^x\, dx = e^x$$

Step 3: Combine.

$$= 5\ln|x| - 3(-\cos(x)) + 4e^x + C$$

$$\boxed{\int \left(\frac{5}{x} - 3\sin(x) + 4e^x\right) dx = 5\ln|x| + 3\cos(x) + 4e^x + C}$$

Example 4 (Splitting with a Root Denominator): Find $\displaystyle\int \frac{6x^3 + 4x}{\sqrt{x}}\, dx$.

Step 1: Rewrite $\sqrt{x} = x^{1/2}$ and split each term.

$$\frac{6x^3}{x^{1/2}} + \frac{4x}{x^{1/2}} = 6x^{5/2} + 4x^{1/2}$$

Step 2: Integrate using the reverse power rule.

$$\int \left(6x^{5/2} + 4x^{1/2}\right) dx = 6 \cdot \frac{x^{7/2}}{7/2} + 4 \cdot \frac{x^{3/2}}{3/2} + C$$

Step 3: Simplify by flipping the fraction divisors.

$$= 6 \cdot \frac{2}{7}x^{7/2} + 4 \cdot \frac{2}{3}x^{3/2} + C = \frac{12}{7}x^{7/2} + \frac{8}{3}x^{3/2} + C$$

$$\boxed{\int \frac{6x^3 + 4x}{\sqrt{x}}\,dx = \frac{12}{7}x^{7/2} + \frac{8}{3}x^{3/2} + C}$$

Example 5 (Connecting to FTC): Use an indefinite integral to evaluate $\displaystyle\int_1^2 \left(6x^2 - \frac{4}{x^2}\right) dx$.

Step 1: Find the indefinite integral first.
Rewrite $\dfrac{4}{x^2} = 4x^{-2}$, then integrate:

$$\int \left(6x^2 - 4x^{-2}\right) dx = 2x^3 - 4 \cdot \frac{x^{-1}}{-1} + C = 2x^3 + \frac{4}{x} + C$$

Step 2: Drop $+C$ and apply FTC Part 2.

$$\int_1^2 \left(6x^2 - \frac{4}{x^2}\right) dx = \left[2x^3 + \frac{4}{x}\right]_1^2$$

Step 3: Evaluate at $x = 2$.

$$2(8) + \frac{4}{2} = 16 + 2 = 18$$

Step 4: Evaluate at $x = 1$.

$$2(1) + \frac{4}{1} = 2 + 4 = 6$$

Step 5: Subtract.

$$18 - 6 = 12$$

$$\boxed{\int_1^2 \left(6x^2 - \frac{4}{x^2}\right) dx = 12}$$

Common Mistakes

- **Splitting a fraction with a sum in the denominator:** You can only split $\dfrac{f(x) + g(x)}{h(x)}$ when $h(x)$ is a *single term*. Never try to split something like $\dfrac{x^2}{x+3}$ — the denominator is a sum, so this technique does not apply.

- **Forgetting $+C$:** Every indefinite integral result must end with $+C$. It represents an entire family of functions, not just one.
- **Leaving dx out of the integral:** The dx is part of the notation and indicates what variable you are integrating. Omitting it is technically incorrect and will cause confusion in later topics like u-substitution.
- **Treating the integral of a product as the product of integrals:** $\int f(x)g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$. Always expand or rewrite a product algebraically before integrating.
- **Forgetting to simplify after splitting:** After dividing each term by the denominator, simplify the resulting powers of x completely before integrating. Carrying unsimplified exponents into the power rule leads to wrong answers.

Worksheet

1. Find $\int (9x^5 - 4x^3 + 7x - 3)\,dx$.

2. Find $\int \left(\frac{1}{2}x^4 - 3x^{2/3} + 5\right)dx$.

3. Find $\int \frac{x^5 + 3x^3 - x}{x^3}\,dx$.

   (Split the fraction first.)

4. Find $\displaystyle\int \frac{4x^3 - 9\sqrt{x}}{x}\,dx$.
   (Split the fraction first, then rewrite any roots.)

5. Find $\displaystyle\int \left(8\cos(x) + \frac{3}{x} - 5x^2\right) dx$.

6. Find $\displaystyle\int \left(2\sec^2(x) - 7e^x + \frac{1}{x}\right) dx$.

7. Find $\displaystyle\int \frac{(x+3)^2}{\sqrt{x}}\,dx$.
   (Expand the numerator first, then split the fraction.)

8. Use an indefinite integral to evaluate $\int_0^3 (4x^3 - 6x)\,dx$.

9. Use an indefinite integral to evaluate $\int_0^{\pi/2} (2\cos(x) + 1)\,dx$.

10. (Challenge) Find $\int \frac{x^4 - 16}{x^2 + 4}\,dx$.
    (Hint: Factor the numerator as a difference of squares, then simplify.)

11. (Challenge) A student claims that $\int \frac{1}{x^2 + 1}\,dx = \ln(x^2 + 1) + C$. Disprove this by differentiating the claimed answer. Then explain what went wrong.

Answer Key

1. $\int (9x^5 - 4x^3 + 7x - 3)\,dx$

Apply the reverse power rule term by term:

$$= \frac{9x^6}{6} - \frac{4x^4}{4} + \frac{7x^2}{2} - 3x + C$$

$$\boxed{\frac{3x^6}{2} - x^4 + \frac{7x^2}{2} - 3x + C}$$

2. $\int \left(\frac{1}{2}x^4 - 3x^{2/3} + 5\right) dx$

$$= \frac{1}{2} \cdot \frac{x^5}{5} - 3 \cdot \frac{x^{5/3}}{5/3} + 5x + C = \frac{x^5}{10} - \frac{9}{5}x^{5/3} + 5x + C$$

$$\boxed{\frac{x^5}{10} - \frac{9}{5}x^{5/3} + 5x + C}$$

3. $\int \frac{x^5 + 3x^3 - x}{x^3}\,dx$

Step 1: Split the fraction.

$$\frac{x^5}{x^3} + \frac{3x^3}{x^3} - \frac{x}{x^3} = x^2 + 3 - x^{-2}$$

Step 2: Integrate.

$$\int (x^2 + 3 - x^{-2})\,dx = \frac{x^3}{3} + 3x - \frac{x^{-1}}{-1} + C$$

$$\boxed{\frac{x^3}{3} + 3x + \frac{1}{x} + C}$$

4. $\int \frac{4x^3 - 9\sqrt{x}}{x}\,dx$

Step 1: Split and simplify. Write $\sqrt{x} = x^{1/2}$.

$$\frac{4x^3}{x} - \frac{9x^{1/2}}{x} = 4x^2 - 9x^{-1/2}$$

Step 2: Integrate.

$$\int (4x^2 - 9x^{-1/2})\,dx = \frac{4x^3}{3} - 9 \cdot \frac{x^{1/2}}{1/2} + C = \frac{4x^3}{3} - 18\sqrt{x} + C$$

$$\boxed{\frac{4x^3}{3} - 18\sqrt{x} + C}$$

5. $\displaystyle\int \left(8\cos(x) + \frac{3}{x} - 5x^2\right) dx$

$$= 8\sin(x) + 3\ln|x| - \frac{5x^3}{3} + C$$

$$\boxed{8\sin(x) + 3\ln|x| - \frac{5x^3}{3} + C}$$

6. $\displaystyle\int \left(2\sec^2(x) - 7e^x + \frac{1}{x}\right) dx$

$$= 2\tan(x) - 7e^x + \ln|x| + C$$

$$\boxed{2\tan(x) - 7e^x + \ln|x| + C}$$

7. $\displaystyle\int \frac{(x+3)^2}{\sqrt{x}}\, dx$
Step 1: Expand $(x+3)^2 = x^2 + 6x + 9$.
Step 2: Divide each term by $x^{1/2}$:

$$\frac{x^2}{x^{1/2}} + \frac{6x}{x^{1/2}} + \frac{9}{x^{1/2}} = x^{3/2} + 6x^{1/2} + 9x^{-1/2}$$

Step 3: Integrate.

$$= \frac{x^{5/2}}{5/2} + 6 \cdot \frac{x^{3/2}}{3/2} + 9 \cdot \frac{x^{1/2}}{1/2} + C = \frac{2}{5}x^{5/2} + 4x^{3/2} + 18\sqrt{x} + C$$

$$\boxed{\frac{2}{5}x^{5/2} + 4x^{3/2} + 18\sqrt{x} + C}$$

8. $\displaystyle\int_0^3 (4x^3 - 6x)\, dx$
Step 1: Find the indefinite integral.

$$\int (4x^3 - 6x)\, dx = x^4 - 3x^2$$

Step 2: Apply FTC Part 2.

$$\left[x^4 - 3x^2\right]_0^3 = \left[(3)^4 - 3(3)^2\right] - [0 - 0] = [81 - 27] - 0 = 54$$

$$\boxed{\int_0^3 (4x^3 - 6x)\, dx = 54}$$

9. $\int_0^{\pi/2} (2\cos(x) + 1)\,dx$

Step 1: Find the indefinite integral.

$$\int (2\cos(x) + 1)\,dx = 2\sin(x) + x$$

Step 2: Apply FTC Part 2.

$$\Big[2\sin(x) + x\Big]_0^{\pi/2} = \left[2\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\right] - [2\sin(0) + 0] = \left[2(1) + \frac{\pi}{2}\right] - 0$$

$$\boxed{\int_0^{\pi/2} (2\cos(x) + 1)\,dx = 2 + \frac{\pi}{2}}$$

10. (Challenge) $\int \frac{x^4 - 16}{x^2 + 4}\,dx$

Step 1: Factor the numerator as a difference of squares.

$$x^4 - 16 = (x^2 + 4)(x^2 - 4)$$

Step 2: Cancel the common factor.

$$\frac{(x^2 + 4)(x^2 - 4)}{x^2 + 4} = x^2 - 4$$

Step 3: Integrate.

$$\int (x^2 - 4)\,dx = \frac{x^3}{3} - 4x + C$$

$$\boxed{\frac{x^3}{3} - 4x + C}$$

11. (Challenge) Disproving the student's claim.

Step 1: Differentiate the claimed answer $\ln(x^2 + 1) + C$ using the chain rule.

$$\frac{d}{dx}\left[\ln(x^2 + 1)\right] = \frac{2x}{x^2 + 1}$$

Step 2: This does *not* equal $\frac{1}{x^2 + 1}$, so the claimed antiderivative is wrong.

Step 3: The student incorrectly applied the rule $\int \frac{1}{x}\,dx = \ln|x|$ by treating $x^2 + 1$ as if it were just x. The denominator is a *sum*, not a single power of x, so the ln rule does not apply here. The correct antiderivative is $\arctan(x) + C$, which comes from an inverse trig rule covered in a later lesson.

$$\boxed{\text{The correct antiderivative is } \arctan(x) + C}$$

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