# Integrals Unit Review
# ApexMath

---

## Part 1: Riemann Approximation

1. Approximate $\int_0^4 (x^2 + 2)\, dx$ using $n = 4$ subintervals with the **Left Sum** $L_4$.

2. Approximate $\int_1^3 \frac{1}{x}\, dx$ using $n = 4$ subintervals with the **Trapezoidal Sum** $T_4$.

3. Approximate $\int_0^2 e^x\, dx$ using $n = 4$ subintervals with the **Midpoint Sum** $M_4$.

## Part 2: Defining the Integral as a Limit of Riemann Sums

4. Write $\int_0^3 (x^2 + 1)\, dx$ as a limit of a Riemann sum using right endpoints. Do not evaluate.

5. Identify the definite integral represented by:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^2 \cdot \frac{3}{n}$$

6. Write $\int_0^2 x^3\, dx$ as a limit of a Riemann sum using right endpoints, then evaluate the limit.

$$\text{Recall: } \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

## Part 3: Fundamental Theorem of Calculus

7. Find $\dfrac{d}{dx}\displaystyle\int_1^x (t^4 - \cos t)\,dt$.

8. Find $\dfrac{d}{dx}\displaystyle\int_0^{x^3} \sin(t^2)\,dt$.

9. Find $\dfrac{d}{dx}\displaystyle\int_x^{x^2} e^t\,dt$.

10. Evaluate $\displaystyle\int_1^4 (3x^2 - 2x + 1)\,dx$.

11. Evaluate $\displaystyle\int_0^{\pi/2} (\sin x + \cos x)\,dx$.

12. Evaluate $\int_1^e \frac{3}{x}\,dx$.

## Part 4: Antiderivatives

13. Find the general antiderivative of $f(x) = 5x^4 - 6x^2 + 3$.

14. Find $\int \left(\sqrt[3]{x^2} + \frac{2}{x^3}\right) dx$.

15. Given $f'(x) = 6x^2 - 4x$ and $f(1) = 3$, find $f(x)$.

16. Given $f''(x) = 4x$, $f'(0) = 1$, and $f(0) = 2$, find $f(x)$.

## Part 5: Indefinite Integrals

17. Find $\int \left(7e^x - 4\cos x + \frac{2}{x}\right) dx$.

18. Find $\int \frac{x^4 - 3x^2 + 1}{x^2}\, dx$.

    *(Split the fraction first.)*

19. Find $\int \frac{(x+2)^2}{\sqrt{x}}\, dx$.

    *(Expand, then split.)*

## Part 6: Properties of Definite Integrals

20. Given $\int_0^5 f(x)\, dx = 12$ and $\int_0^5 g(x)\, dx = -3$, find $\int_0^5 [3f(x) - 2g(x)]\, dx$.

21. Given $\int_1^7 f(x)\,dx = 10$ and $\int_4^7 f(x)\,dx = 6$, find $\int_1^4 f(x)\,dx$.

22. Suppose $1 \leq f(x) \leq 4$ for all $x$ in $[2, 6]$. Use the bounding property to give upper and lower bounds for $\int_2^6 f(x)\,dx$.

## Part 7: U-Substitution

23. Find $\int (4x+3)^5\,dx$.

24. Find $\int x^2 \cos(x^3)\,dx$.

25. Find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

26. Evaluate $\int_0^2 x(x^2+1)^3\,dx$.

## Part 8: Integration Using Trig Identities

27. Find $\int \cos^2(x)\,dx$.

28. Find $\int \sin^3(x)\cos^2(x)\,dx$.

29. Find $\int \tan^2(x)\,dx$.

## Part 9: Integration by Parts

30. Find $\displaystyle\int x \sin(x)\, dx$.

31. Find $\displaystyle\int x^2 e^x\, dx$.

32. Find $\displaystyle\int \ln(x)\, dx$.

33. Evaluate $\displaystyle\int_1^e x \ln(x)\, dx$.

## Part 10: Trigonometric Substitution

34. Find $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx.$

35. Find $\displaystyle\int \sqrt{9-x^2}\,dx.$

36. Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$

## Part 11: Partial Fraction Decomposition

37. Find $\displaystyle\int \frac{4x+1}{(x-1)(x+3)}\,dx.$

38. Find $\displaystyle\int \frac{2x+5}{(x+1)^2}\,dx$.

39. Find $\displaystyle\int \frac{x^2+2}{x(x^2+1)}\,dx$.

# Part 12: Improper Integrals

40. Evaluate $\displaystyle\int_2^\infty \frac{1}{x^2}\,dx$.

41. Evaluate $\displaystyle\int_{-\infty}^0 e^{3x}\,dx$.

42. Evaluate $\displaystyle\int_0^1 \frac{1}{x^{1/3}}\,dx$.

43. Evaluate $\displaystyle\int_0^4 \frac{1}{(x-2)^2}\,dx$.

*(Asymptote at interior point $x = 2$. Split there.)*

# Answer Key — Integrals Unit Review
# ApexMath

---

## Part 1: Riemann Approximation

**1.** $L_4$ for $\displaystyle\int_0^4 (x^2+2)\,dx$, $n=4$.
$\Delta x = 1$. Endpoints: $x_0=0, x_1=1, x_2=2, x_3=3, x_4=4$.
$f(0)=2,\ f(1)=3,\ f(2)=6,\ f(3)=11$.
$L_4 = 1[2+3+6+11]$

$$\boxed{L_4 = 22}$$

**2.** $T_4$ for $\displaystyle\int_1^3 \frac{1}{x}\,dx$, $n=4$.
$\Delta x = 0.5$. Endpoints: $1, 1.5, 2, 2.5, 3$.
$f(1)=1,\ f(1.5)\approx 0.6667,\ f(2)=0.5,\ f(2.5)=0.4,\ f(3)\approx 0.3333$.

$$T_4 = \frac{0.5}{2}[1+2(0.6667)+2(0.5)+2(0.4)+0.3333] = 0.25[1+1.3334+1+0.8+0.3333] = 0.25(4.4667)$$

$$\boxed{T_4 \approx 1.1167}$$

**3.** $M_4$ for $\displaystyle\int_0^2 e^x\,dx$, $n=4$.
$\Delta x = 0.5$. Midpoints: $0.25, 0.75, 1.25, 1.75$.
$f(0.25)\approx 1.2840,\ f(0.75)\approx 2.1170,\ f(1.25)\approx 3.4903,\ f(1.75)\approx 5.7546$.
$M_4 = 0.5[1.2840+2.1170+3.4903+5.7546] = 0.5(12.6459)$

$$\boxed{M_4 \approx 6.3230}$$

## Part 2: Riemann Sums and the Integral

**4.** $\Delta x = \dfrac{3}{n}$, $x_i = \dfrac{3i}{n}$, $f(x_i) = \dfrac{9i^2}{n^2}+1$.

$$\boxed{\int_0^3 (x^2+1)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}+1\right)\frac{3}{n}}$$

**5.** $\Delta x = \frac{3}{n}$, $x_i = 2 + \frac{3i}{n}$, $f(x_i) = \left(2 + \frac{3i}{n}\right)^2$. Lower limit $a = 2$, upper limit $b = 5$.

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^2 \cdot \frac{3}{n} = \int_2^5 x^2\,dx}$$

**6.** $\Delta x = \frac{2}{n}$, $x_i = \frac{2i}{n}$, $f(x_i) = \frac{8i^3}{n^3}$.

$$\lim_{n\to\infty} \frac{16}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} = \lim_{n\to\infty} \frac{4(n+1)^2}{n^2} = 4$$

$$\boxed{\int_0^2 x^3\,dx = 4}$$

## Part 3: Fundamental Theorem of Calculus

**7.** FTC Part 1 directly: replace $t$ with $x$.

$$\boxed{\frac{d}{dx} \int_1^x (t^4 - \cos t)\,dt = x^4 - \cos x}$$

**8.** Upper limit $g(x) = x^3$, $g'(x) = 3x^2$. Apply chain rule.

$$\boxed{\frac{d}{dx} \int_0^{x^3} \sin(t^2)\,dt = \sin(x^6) \cdot 3x^2}$$

**9.** Split at constant $c$: $\int_x^{x^2} = \int_c^{x^2} - \int_c^x$.

$$\frac{d}{dx} \int_c^{x^2} e^t\,dt - \frac{d}{dx} \int_c^x e^t\,dt = e^{x^2} \cdot 2x - e^x$$

$$\boxed{\frac{d}{dx} \int_x^{x^2} e^t\,dt = 2xe^{x^2} - e^x}$$

**10.** $F(x) = x^3 - x^2 + x$.

$$\Big[x^3 - x^2 + x\Big]_1^4 = (64 - 16 + 4) - (1 - 1 + 1) = 52 - 1$$

$$\boxed{\int_1^4 (3x^2 - 2x + 1)\,dx = 51}$$

**11.** $F(x) = -\cos x + \sin x$.

$$[-\cos x + \sin x]_0^{\pi/2} = (-\cos(\pi/2) + \sin(\pi/2)) - (-\cos 0 + \sin 0) = (0+1) - (-1+0) = 1+1$$

$$\boxed{\int_0^{\pi/2} (\sin x + \cos x)\,dx = 2}$$

**12.** $F(x) = 3\ln|x|$.

$$\Big[3\ln x\Big]_1^e = 3\ln e - 3\ln 1 = 3(1) - 3(0) = 3$$

$$\boxed{\int_1^e \frac{3}{x}\,dx = 3}$$

## Part 4: Antiderivatives

**13.** Apply reverse power rule term by term.

$$\boxed{F(x) = x^5 - 2x^3 + 3x + C}$$

**14.** Rewrite: $x^{2/3} + 2x^{-3}$.

$$\frac{x^{5/3}}{5/3} + \frac{2x^{-2}}{-2} + C = \frac{3}{5}x^{5/3} - \frac{1}{x^2} + C$$

$$\boxed{\frac{3}{5}x^{5/3} - \frac{1}{x^2} + C}$$

**15.** $f(x) = 2x^3 - 2x^2 + C$. Use $f(1) = 3$: $2 - 2 + C = 3 \Rightarrow C = 3$.

$$\boxed{f(x) = 2x^3 - 2x^2 + 3}$$

**16.** Integrate twice.
$f'(x) = \int 4x\,dx = 2x^2 + C_1$. Use $f'(0) = 1$: $C_1 = 1$. So $f'(x) = 2x^2 + 1$.
$f(x) = \int (2x^2 + 1)\,dx = \dfrac{2x^3}{3} + x + C_2$. Use $f(0) = 2$: $C_2 = 2$.

$$\boxed{f(x) = \frac{2x^3}{3} + x + 2}$$

## Part 5: Indefinite Integrals

**17.**

$$\boxed{7e^x - 4\sin x + 2\ln|x| + C}$$

**18.** Split: $x^2 - 3 + x^{-2}$.

$$\frac{x^3}{3} - 3x - \frac{1}{x} + C$$

$$\boxed{\frac{x^3}{3} - 3x - \frac{1}{x} + C}$$

**19.** Expand $(x+2)^2 = x^2 + 4x + 4$. Divide by $x^{1/2}$: $x^{3/2} + 4x^{1/2} + 4x^{-1/2}$.

$$\frac{x^{5/2}}{5/2} + 4 \cdot \frac{x^{3/2}}{3/2} + 4 \cdot \frac{x^{1/2}}{1/2} + C = \frac{2}{5}x^{5/2} + \frac{8}{3}x^{3/2} + 8x^{1/2} + C$$

$$\boxed{\frac{2}{5}x^{5/2} + \frac{8}{3}x^{3/2} + 8\sqrt{x} + C}$$

## Part 6: Properties of Definite Integrals

**20.** $3(12) - 2(-3) = 36 + 6$

$$\boxed{\int_0^5 [3f - 2g]\,dx = 42}$$

**21.** Additivity: $\displaystyle\int_1^7 = \int_1^4 + \int_4^7$, so $10 = \int_1^4 + 6$.

$$\boxed{\int_1^4 f(x)\,dx = 4}$$

**22.** $m = 1$, $M = 4$, $b - a = 4$.

$$\boxed{4 \le \int_2^6 f(x)\,dx \le 16}$$

## Part 7: U-Substitution

**23.** $u = 4x + 3$, $du = 4\,dx$, $dx = \frac{du}{4}$.

$$\frac{1}{4} \cdot \frac{u^6}{6} + C = \frac{(4x+3)^6}{24} + C$$

$$\boxed{\frac{(4x+3)^6}{24} + C}$$

**24.** $u = x^3$, $du = 3x^2\,dx$, $x^2\,dx = \frac{du}{3}$.

$$\frac{1}{3}\sin(u) + C$$

$$\boxed{\frac{\sin(x^3)}{3} + C}$$

**25.** $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, $\frac{dx}{\sqrt{x}} = 2\,du$.

$$2\int e^u\,du = 2e^u + C$$

$$\boxed{2e^{\sqrt{x}} + C}$$

**26.** $u = x^2 + 1$, $du = 2x\,dx$. Limits: $x = 0 \Rightarrow u = 1$; $x = 2 \Rightarrow u = 5$.

$$\frac{1}{2}\int_1^5 u^3\,du = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^5 = \frac{1}{2} \cdot \frac{625-1}{4} = \frac{624}{8} = 78$$

$$\boxed{\int_0^2 x(x^2+1)^3\,dx = 78}$$

## Part 8: Trig Identities

**27.** Use $\cos^2 x = \dfrac{1+\cos 2x}{2}$:

$$\frac{1}{2}\int (1+\cos 2x)\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

$$\boxed{\frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

**28.** Odd sin power. Save $\sin x$, convert $\sin^2 x = 1-\cos^2 x$. Let $u = \cos x$, $du = -\sin x\,dx$:

$$-\int (1-u^2)u^2\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$\boxed{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C}$$

**29.** Use $\tan^2 x = \sec^2 x - 1$:

$$\int (\sec^2 x - 1)\,dx = \tan x - x + C$$

$$\boxed{\tan x - x + C}$$

## Part 9: Integration by Parts

**30.** $u = x$, $dv = \sin x\,dx$, $v = -\cos x$.

$$-x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

$$\boxed{-x\cos x + \sin x + C}$$

**31.** Two applications of IBP.
First: $u = x^2$, $dv = e^x\,dx$: $x^2e^x - 2\int xe^x\,dx$.
Second: $u = x$, $dv = e^x\,dx$: $xe^x - e^x$.

$$x^2e^x - 2(xe^x - e^x) + C$$

$$\boxed{x^2e^x - 2xe^x + 2e^x + C}$$

**32.** $u = \ln x$, $dv = dx$, $du = \frac{1}{x}\,dx$, $v = x$.

$$x\ln x - \int 1\,dx = x\ln x - x + C$$

$$\boxed{x\ln x - x + C}$$

**33.** $u = \ln x$, $dv = x\,dx$, $v = \frac{x^2}{2}$.

$$\left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x}{2}\,dx = \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \frac{e^2-1}{4} = \frac{2e^2-(e^2-1)}{4}$$

$$\boxed{\int_1^e x\ln x\,dx = \frac{e^2+1}{4}}$$

## Part 10: Trig Substitution

**34.** $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1-x^2} = \cos\theta$.

$$\int \frac{\cos\theta}{\cos\theta}\, d\theta = \theta + C = \arcsin(x) + C$$

$$\boxed{\arcsin(x) + C}$$

**35.** $a = 3$, $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$, $\sqrt{9-x^2} = 3\cos\theta$.

$$\int 3\cos\theta \cdot 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta = \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C = \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C$$

Reference triangle: $\sin\theta = \frac{x}{3}$, $\cos\theta = \frac{\sqrt{9-x^2}}{3}$, $\theta = \arcsin\left(\frac{x}{3}\right)$.

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$\boxed{\frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C}$$

**36.** $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $\sqrt{x^2+4} = 2\sec\theta$.

$$\int \frac{2\sec^2\theta}{4\tan^2\theta \cdot 2\sec\theta}\, d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\int \csc\theta\cot\theta\, d\theta = -\frac{\csc\theta}{4} + C$$

Reference triangle: $\tan\theta = \frac{x}{2}$, $\csc\theta = \frac{\sqrt{x^2+4}}{x}$.

$$\boxed{\int \frac{1}{x^2\sqrt{x^2+4}}\, dx = -\frac{\sqrt{x^2+4}}{4x} + C}$$

## Part 11: Partial Fraction Decomposition

**37.** $\dfrac{4x+1}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$.

$4x + 1 = A(x+3) + B(x-1)$.

$x = 1$: $5 = 4A \Rightarrow A = \frac{5}{4}$. $x = -3$: $-11 = -4B \Rightarrow B = \frac{11}{4}$.

$$\boxed{\frac{5}{4}\ln|x-1| + \frac{11}{4}\ln|x+3| + C}$$

**38.** $\dfrac{2x+5}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$.

$2x+5 = A(x+1)+B$. $x=-1$: $3 = B$. Match $x$: $A = 2$.

$$2\ln|x+1| + \frac{3(x+1)^{-1}}{-1}\cdot(-1) + C$$

$\int \frac{3}{(x+1)^2}\,dx = -\frac{3}{x+1}$

$$\boxed{2\ln|x+1| - \frac{3}{x+1} + C}$$

**39.** $\dfrac{x^2+2}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$.
$x^2+2 = A(x^2+1) + (Bx+C)x$.
$x=0$: $2 = A$. Match $x^2$: $1 = A+B \Rightarrow B = -1$. Match $x^1$: $0 = C$.

$$\int\left(\frac{2}{x} - \frac{x}{x^2+1}\right)dx = 2\ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

$$\boxed{2\ln|x| - \frac{1}{2}\ln(x^2+1) + C}$$

## Part 12: Improper Integrals

**40.**

$$\lim_{t\to\infty}\left[-\frac{1}{x}\right]_2^t = \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{2}\right) = \frac{1}{2}$$

$$\boxed{\int_2^\infty \frac{1}{x^2}\,dx = \frac{1}{2}}$$

**41.**

$$\lim_{t\to-\infty}\left[\frac{e^{3x}}{3}\right]_t^0 = \lim_{t\to-\infty}\left(\frac{1}{3}-\frac{e^{3t}}{3}\right) = \frac{1}{3} - 0$$

$$\boxed{\int_{-\infty}^0 e^{3x}\,dx = \frac{1}{3}}$$

**42.** Asymptote at $x = 0$ (lower limit).

$$\lim_{t\to0^+}\left[\frac{x^{2/3}}{2/3}\right]_t^1 = \lim_{t\to0^+}\left(\frac{3}{2}-\frac{3}{2}t^{2/3}\right) = \frac{3}{2}$$

$$\boxed{\int_0^1 \frac{1}{x^{1/3}}\,dx = \frac{3}{2}}$$

**43.** Asymptote at $x = 2$ (inside $[0, 4]$). Split there.
Antiderivative: $\int (x-2)^{-2}\,dx = -\dfrac{1}{x-2}$.
Left piece (0 to $2^-$):

$$\lim_{t\to 2^-}\left[-\frac{1}{x-2}\right]_0^t = \lim_{t\to 2^-}\left(-\frac{1}{t-2}+\frac{1}{-2}\right)$$

As $t \to 2^-$, $-\dfrac{1}{t-2} \to +\infty$. The left piece **diverges**.
Since one piece diverges, the entire integral diverges.

$$\boxed{\int_0^4 \frac{1}{(x-2)^2}\,dx \text{ diverges}}$$

---

# Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

*End of Answer Key • ApexMath*

---