Integrating Using Trig Identities
ApexMath

Notes

Why Trig Identities?

Many integrals involving trigonometric functions cannot be solved directly with the basic antiderivative rules. The integrand needs to be **rewritten first** using a trig identity before integration becomes possible. The identities transform powers and products of trig functions into simpler forms that we already know how to integrate.

The Key Identities to Know

Pythagorean Identities:

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin^2(x) = 1 - \cos^2(x) \qquad \cos^2(x) = 1 - \sin^2(x)$$

$$1 + \tan^2(x) = \sec^2(x) \qquad \tan^2(x) = \sec^2(x) - 1$$

$$1 + \cot^2(x) = \csc^2(x) \qquad \cot^2(x) = \csc^2(x) - 1$$

Power-Reducing (Half-Angle) Identities:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \qquad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

These are essential for integrating even powers of sine and cosine.

Double Angle Identity:

$$\sin(2x) = 2\sin(x)\cos(x) \implies \sin(x)\cos(x) = \frac{\sin(2x)}{2}$$

Strategy for Powers of Sine and Cosine: $\displaystyle\int \sin^m(x)\cos^n(x)\,dx$

Case 1 — At least one power is odd.

Save one factor of the odd-powered function to pair with dx for a u-substitution. Convert all remaining even powers using the Pythagorean identity.

For example, if n is odd, save one $\cos(x)$ and convert $\cos^{n-1}(x)$ using $\cos^2(x) = 1 - \sin^2(x)$. Then let $u = \sin(x)$.

Case 2 — Both powers are even.

Use the power-reducing identities to reduce all exponents to 1. This may need to be applied more than once if the result still has even powers.

Strategy for Powers of Tangent and Secant

- To integrate $\tan^n(x)$: use $\tan^2(x) = \sec^2(x) - 1$ to peel off one $\tan^2$ at a time.
- To integrate $\sec^n(x)$ with even n: save $\sec^2(x)$ for du and convert the rest using $\sec^2(x) = 1 + \tan^2(x)$. Let $u = \tan(x)$.
- The integral $\displaystyle\int \tan(x)\,dx = \ln|\sec(x)| + C$ and $\displaystyle\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$ are standard results worth memorizing.

Pro Tip

- Before integrating any power of sine or cosine, immediately classify it as odd or even. This determines your entire strategy before you write a single line of work.
- When both powers are even, the power-reducing identities will introduce a $\cos(2x)$ term. Do not panic — integrate $\cos(2x)$ using a quick u-sub: $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$.
- The odd-power strategy always works cleanly because saving one factor leaves an even power, which rewrites perfectly using $\sin^2 = 1 - \cos^2$ (or vice versa).
- For $\tan^2(x)$, the substitution $\tan^2(x) = \sec^2(x) - 1$ is almost always the first move. The $\sec^2(x)$ integrates directly to $\tan(x)$ and the 1 integrates to x.
- Always expand fully after substituting the identity before integrating. Trying to integrate before simplifying leads to errors.

Examples

Example 1 (Even Power — Power-Reducing): Find $\displaystyle\int \cos^2(x)\,dx$.

Step 1: Both powers are even ($m = 0$, $n = 2$). Use the power-reducing identity.

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Step 2: Rewrite the integral.

$$\int \cos^2(x)\,dx = \int \frac{1 + \cos(2x)}{2}\,dx = \frac{1}{2}\int (1 + \cos(2x))\,dx$$

Step 3: Integrate term by term. For $\cos(2x)$, use a quick u-sub with $u = 2x$, giving $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x)$.

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right] + C$$

Step 4: Distribute the $\frac{1}{2}$.

$$\boxed{\int \cos^2(x)\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

Example 2 (Odd Power): Find $\displaystyle\int \sin^3(x)\,dx$.

Step 1: The power is odd. Save one $\sin(x)$ and convert the rest.

$$\sin^3(x) = \sin^2(x)\cdot\sin(x) = (1-\cos^2(x))\sin(x)$$

Step 2: Rewrite the integral.

$$\int \sin^3(x)\,dx = \int (1-\cos^2(x))\sin(x)\,dx$$

Step 3: Let $u = \cos(x)$, so $du = -\sin(x)\,dx$, meaning $\sin(x)\,dx = -du$.

$$= \int (1-u^2)(-du) = -\int (1-u^2)\,du = \int (u^2-1)\,du$$

Step 4: Integrate.

$$= \frac{u^3}{3} - u + C$$

Step 5: Substitute back $u = \cos(x)$.

$$\boxed{\int \sin^3(x)\,dx = \frac{\cos^3(x)}{3} - \cos(x) + C}$$

Example 3 (Mixed Powers — Odd): Find $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$.

Step 1: $\cos$ has an odd power. Save one $\cos(x)$ and convert $\cos^2(x)$.

$$\sin^2(x)\cos^3(x) = \sin^2(x)\cos^2(x)\cdot\cos(x) = \sin^2(x)(1-\sin^2(x))\cos(x)$$

Step 2: Let $u = \sin(x)$, so $du = \cos(x)\,dx$.

$$\int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx = \int u^2(1-u^2)\,du$$

Step 3: Expand.

$$= \int (u^2 - u^4)\,du$$

Step 4: Integrate.

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

Step 5: Substitute back.

$$\boxed{\int \sin^2(x)\cos^3(x)\,dx = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$$

Example 4 (Tangent Power): Find $\displaystyle\int \tan^2(x)\,dx$.

Step 1: Use the Pythagorean identity $\tan^2(x) = \sec^2(x) - 1$**.**

$$\int \tan^2(x)\,dx = \int (\sec^2(x) - 1)\,dx$$

Step 2: Integrate term by term.

$$= \int \sec^2(x)\,dx - \int 1\,dx = \tan(x) - x + C$$

$$\boxed{\int \tan^2(x)\,dx = \tan(x) - x + C}$$

Example 5 (Definite Integral with Trig Identity): Evaluate $\displaystyle\int_0^{\pi/2} \sin^2(x)\,dx$.

Step 1: Both powers even. Use the power-reducing identity.

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Step 2: Rewrite the integral.

$$\int_0^{\pi/2} \sin^2(x)\,dx = \frac{1}{2}\int_0^{\pi/2} (1 - \cos(2x))\,dx$$

Step 3: Find the antiderivative. Use $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x)$**.**

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi/2}$$

Step 4: Evaluate at $x = \pi/2$**.**

$$\frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin(\pi)\right] = \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}(0)\right] = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

Step 5: Evaluate at $x = 0$.

$$\frac{1}{2}\left[0 - \frac{1}{2}\sin(0)\right] = \frac{1}{2}(0) = 0$$

Step 6: Subtract.

$$\frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\boxed{\int_0^{\pi/2} \sin^2(x)\,dx = \frac{\pi}{4}}$$

Common Mistakes

- **Using the wrong power-reducing identity:** $\sin^2(x)$ uses $\dfrac{1-\cos(2x)}{2}$ and $\cos^2(x)$ uses $\dfrac{1+\cos(2x)}{2}$. The signs are opposite — sin gets the minus, cos gets the plus. Mixing these up flips the entire answer.

- **Forgetting to save one factor for the odd-power strategy:** You must save one $\sin(x)$ or $\cos(x)$ to become du in the substitution. Converting *all* the factors with the Pythagorean identity and having nothing left for du is a very common error.

- **Not expanding after substituting the identity:** After applying $1 - \cos^2(x)$ or $1 - \sin^2(x)$, you must multiply through and expand before integrating. Integrating an unexpanded product directly is incorrect.

- **Forgetting the $\frac{1}{2}$ when integrating $\cos(2x)$:** Since $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$ (from a quick u-sub), missing the $\frac{1}{2}$ is a frequent arithmetic slip in even-power problems.

- **Applying the odd-power strategy when both powers are even:** If *both* m and n are even, there is no factor to save. You must use the power-reducing identities instead.

Worksheet

1. Find $\displaystyle\int \sin^2(x)\,dx$.

2. Find $\int \cos^3(x)\,dx$.

3. Find $\int \sin^3(x)\cos^2(x)\,dx$.

4. Find $\int \cos^4(x)\,dx$.

   (Apply the power-reducing identity twice.)

5. Find $\int \tan^3(x)\,dx$.

   Hint: Write $\tan^3(x) = \tan(x)\cdot\tan^2(x)$, *then use* $\tan^2(x) = \sec^2(x) - 1$.

6. Find $\int \sin(x)\cos(x)\,dx$.

   (Hint: Use the double-angle identity $\sin(x)\cos(x) = \frac{\sin(2x)}{2}$*.)*

7. Find $\int \sec^4(x)\,dx$.

   Hint: Save $\sec^2(x)$ *for du, convert* $\sec^2(x) = 1 + \tan^2(x)$*, let* $u = \tan(x)$.

8. Evaluate $\int_0^{\pi} \cos^2(x)\,dx$.

9. Evaluate $\int_0^{\pi/2} \sin^3(x)\,dx$.

10. (Challenge) Find $\displaystyle\int \sin^4(x)\,dx$.

(Apply power-reducing to $\sin^2(x)$*, square the result, then apply power-reducing again to the* $\cos^2(2x)$ *term.)*

11. (Challenge) Find $\displaystyle\int \sin^2(x)\cos^2(x)\,dx$.

(Hint: Use the double-angle identity $\sin(x)\cos(x) = \frac{\sin(2x)}{2}$ *first.)*

Answer Key

1. $\int \sin^2(x)\,dx$

Use the power-reducing identity $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$:

$$= \frac{1}{2}\int (1-\cos(2x))\,dx = \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C$$

$$\boxed{\int \sin^2(x)\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C}$$

2. $\int \cos^3(x)\,dx$

Odd power. Save one $\cos(x)$ and convert $\cos^2(x) = 1 - \sin^2(x)$:

$$\int (1-\sin^2(x))\cos(x)\,dx$$

Let $u = \sin(x)$, $du = \cos(x)\,dx$:

$$\int (1-u^2)\,du = u - \frac{u^3}{3} + C$$

$$\boxed{\int \cos^3(x)\,dx = \sin(x) - \frac{\sin^3(x)}{3} + C}$$

3. $\int \sin^3(x)\cos^2(x)\,dx$

sin has odd power. Save one $\sin(x)$, convert $\sin^2(x) = 1 - \cos^2(x)$:

$$\int (1-\cos^2(x))\cos^2(x)\sin(x)\,dx$$

Let $u = \cos(x)$, $du = -\sin(x)\,dx$:

$$-\int (1-u^2)u^2\,du = -\int (u^2-u^4)\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$\boxed{\int \sin^3(x)\cos^2(x)\,dx = -\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C}$$

4. $\int \cos^4(x)\,dx$

Both powers even. Write $\cos^4(x) = (\cos^2(x))^2$ and apply $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$:

$$\cos^4(x) = \left(\frac{1+\cos(2x)}{2}\right)^2 = \frac{1+2\cos(2x)+\cos^2(2x)}{4}$$

Apply power-reducing to $\cos^2(2x) = \dfrac{1+\cos(4x)}{2}$:

$$= \frac{1}{4}\left[1+2\cos(2x)+\frac{1+\cos(4x)}{2}\right] = \frac{1}{4}\left[\frac{3}{2}+2\cos(2x)+\frac{\cos(4x)}{2}\right]$$

Integrate:

$$= \frac{1}{4}\left[\frac{3x}{2}+\sin(2x)+\frac{\sin(4x)}{8}\right]+C$$

$$\boxed{\int \cos^4(x)\,dx = \frac{3x}{8}+\frac{\sin(2x)}{4}+\frac{\sin(4x)}{32}+C}$$

5. $\int \tan^3(x)\,dx$

Write $\tan^3(x) = \tan(x)\cdot\tan^2(x) = \tan(x)(\sec^2(x)-1)$:

$$\int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx$$

For the first integral, let $u = \tan(x)$, $du = \sec^2(x)\,dx$:

$$\int u\,du = \frac{u^2}{2} = \frac{\tan^2(x)}{2}$$

For the second integral, recall $\int \tan(x)\,dx = \ln|\sec(x)|+C$:

$$\int \tan^3(x)\,dx = \frac{\tan^2(x)}{2} - \ln|\sec(x)|+C$$

$$\boxed{\int \tan^3(x)\,dx = \frac{\tan^2(x)}{2} - \ln|\sec(x)|+C}$$

6. $\int \sin(x)\cos(x)\,dx$

Use the double-angle identity $\sin(x)\cos(x) = \dfrac{\sin(2x)}{2}$:

$$\int \frac{\sin(2x)}{2}\,dx = \frac{1}{2}\cdot\left(-\frac{\cos(2x)}{2}\right)+C = -\frac{\cos(4x)}{4}+C$$

$$\boxed{\int \sin(x)\cos(x)\,dx = -\frac{\cos(2x)}{4} + C}$$

Note: An equally valid answer via u-sub ($u = \sin x$) is $\frac{\sin^2(x)}{2} + C$*. Both are correct and differ only by a constant.*

7. $\int \sec^4(x)\,dx$

Save $\sec^2(x)$ and convert the remaining $\sec^2(x)$ using $\sec^2(x) = 1 + \tan^2(x)$:

$$\int (1 + \tan^2(x))\sec^2(x)\,dx$$

Let $u = \tan(x)$, $du = \sec^2(x)\,dx$:

$$\int (1 + u^2)\,du = u + \frac{u^3}{3} + C$$

$$\boxed{\int \sec^4(x)\,dx = \tan(x) + \frac{\tan^3(x)}{3} + C}$$

8. $\int_0^{\pi} \cos^2(x)\,dx$

Use $\cos^2(x) = \frac{1 + \cos(2x)}{2}$:

$$\frac{1}{2}\int_0^{\pi} (1 + \cos(2x))\,dx = \frac{1}{2}\left[x + \frac{\sin(2x)}{2}\right]_0^{\pi}$$

Evaluate at $x = \pi$: $\frac{1}{2}\left[\pi + \frac{\sin(2\pi)}{2}\right] = \frac{1}{2}[\pi + 0] = \frac{\pi}{2}$

Evaluate at $x = 0$: $\frac{1}{2}\left[0 + \frac{\sin(0)}{2}\right] = 0$

$$\boxed{\int_0^{\pi} \cos^2(x)\,dx = \frac{\pi}{2}}$$

9. $\int_0^{\pi/2} \sin^3(x)\,dx$

Odd power. Save one $\sin(x)$, convert $\sin^2(x) = 1 - \cos^2(x)$:

$$\int_0^{\pi/2} (1 - \cos^2(x))\sin(x)\,dx$$

Let $u = \cos(x)$, $du = -\sin(x)\,dx$.

Change limits: $x = 0 \Rightarrow u = 1; \quad x = \pi/2 \Rightarrow u = 0.$

$$-\int_1^0 (1-u^2)\,du = \int_0^1 (1-u^2)\,du = \left[u - \frac{u^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\boxed{\int_0^{\pi/2} \sin^3(x)\,dx = \frac{2}{3}}$$

10. (Challenge) $\displaystyle\int \sin^4(x)\,dx$

Step 1: Write $\sin^4(x) = (\sin^2(x))^2$ and apply $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$:

$$\sin^4(x) = \left(\frac{1-\cos(2x)}{2}\right)^2 = \frac{1 - 2\cos(2x) + \cos^2(2x)}{4}$$

Step 2: Apply power-reducing to $\cos^2(2x) = \dfrac{1+\cos(4x)}{2}$:

$$= \frac{1}{4}\left[1 - 2\cos(2x) + \frac{1+\cos(4x)}{2}\right] = \frac{1}{4}\left[\frac{3}{2} - 2\cos(2x) + \frac{\cos(4x)}{2}\right]$$

Step 3: Integrate:

$$= \frac{1}{4}\left[\frac{3x}{2} - \sin(2x) + \frac{\sin(4x)}{8}\right] + C$$

$$\boxed{\int \sin^4(x)\,dx = \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C}$$

11. (Challenge) $\displaystyle\int \sin^2(x)\cos^2(x)\,dx$

Step 1: Use the double-angle identity. Since $\sin(x)\cos(x) = \dfrac{\sin(2x)}{2}$:

$$\sin^2(x)\cos^2(x) = (\sin(x)\cos(x))^2 = \left(\frac{\sin(2x)}{2}\right)^2 = \frac{\sin^2(2x)}{4}$$

Step 2: Apply power-reducing to $\sin^2(2x) = \dfrac{1-\cos(4x)}{2}$:

$$= \frac{1}{4} \cdot \frac{1-\cos(4x)}{2} = \frac{1-\cos(4x)}{8}$$

Step 3: Integrate:

$$\int \frac{1-\cos(4x)}{8}\,dx = \frac{1}{8}\left[x - \frac{\sin(4x)}{4}\right] + C$$

$$\boxed{\int \sin^2(x)\cos^2(x)\,dx = \frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

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