Integration by Parts
ApexMath

Notes

Where Does Integration by Parts Come From?

Integration by parts is the integration technique that reverses the **product rule** for derivatives. Recall the product rule:

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x:

$$uv = \int u\,dv + \int v\,du$$

Rearranging gives the **integration by parts formula**:

$$\int u\,dv = uv - \int v\,du$$

In plain words: the integral of $u\,dv$ equals u times v, minus the integral of $v\,du$. The goal is to choose u and dv so that the new integral $\int v\,du$ is *simpler* than the original.

The LIPET Rule for Choosing u

The hardest part of integration by parts is deciding which factor becomes u and which becomes dv. The **LIPET** rule gives the priority order for choosing u:

Letter	**Function Type**
L	Logarithms: $\ln(x)$, $\log(x)$
I	Inverse trig: $\arctan(x)$, $\arcsin(x)$
P	Polynomials: x^n, constants
E	Exponentials: e^x, a^x
T	Trigonometric: $\sin(x)$, $\cos(x)$

Choose the function that appears **earliest** in LIPET as your u. The remaining factor (together with dx) becomes dv.

For example, in $\int xe^x\,dx$: x is a polynomial (P) and e^x is exponential (E). Since P comes before E in LIPET, choose $u = x$ and $dv = e^x\,dx$.

The Setup Table

It helps to organize the four pieces in a small table before applying the formula:

$u =$	$dv =$
$du =$	$v =$

Fill in u and dv first. Then differentiate u to get du, and *integrate* dv to get v. When finding v, do not add $+C$ — the constant is handled at the very end.

Repeated Integration by Parts

Sometimes after one application of IBP, the new integral $\int v\,du$ still requires IBP again. This happens when u is a polynomial of degree 2 or higher multiplied by an exponential or trig function. Simply apply IBP a second (or third) time to the new integral.

The Loop Trick

When the integrand involves a product of e^x and a trig function (like $e^x \sin(x)$), applying IBP twice will cycle back to the *original integral*. This is not a mistake — it is an opportunity. When the original integral reappears on the right side, treat it as an unknown, call it I, and solve the resulting equation algebraically for I.

For example, if after two applications you arrive at:

$$I = (\text{some expression}) - I$$

then add I to both sides:

$$2I = (\text{some expression}) \implies I = \frac{\text{some expression}}{2}$$

Pro Tip

- After setting up your table, double-check by asking: is $\int v\,du$ simpler than the original? If your dv choice made things harder, swap u and dv and try again.
- For $\int \ln(x)\,dx$, there is only one factor. Set $u = \ln(x)$ and $dv = dx$ (i.e. $v = x$). This is a very common exam question.
- In the loop trick, always apply IBP the *same way* both times. If you swap which function is u on the second application, the two integrals will cancel rather than combine, giving $0 = 0$ — which is useless.
- When doing repeated IBP, the polynomial degree drops by 1 each time. So $\int x^3 e^x\,dx$ will need IBP applied three times.
- For a definite integral with IBP, apply the formula as $\Big[uv\Big]_a^b - \displaystyle\int_a^b v\,du$. Evaluate the uv term at both limits just like FTC.

Examples

Example 1 (Basic IBP): Find $\displaystyle\int xe^x\,dx$.

Step 1: Apply LIPET. x **is a polynomial (P),** e^x **is exponential (E). Choose** $u = x$**.**

$u = x$	$dv = e^x\,dx$
$du = dx$	$v = e^x$

Step 2: Apply the formula $\int u\,dv = uv - \int v\,du$**.**

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

Step 3: Integrate the remaining integral.

$$= xe^x - e^x + C$$

$$\boxed{\int xe^x\,dx = xe^x - e^x + C}$$

Verify: $\dfrac{d}{dx}[xe^x - e^x] = e^x + xe^x - e^x = xe^x$ ✓

Example 2 (Logarithm): Find $\displaystyle\int \ln(x)\,dx$.

Step 1: There is only one factor. By LIPET, $\ln(x)$ **is a logarithm (L) and goes first. Set** $dv = dx$**.**

$u = \ln(x)$	$dv = dx$
$du = \frac{1}{x}\,dx$	$v = x$

Step 2: Apply the formula.

$$\int \ln(x)\,dx = x\ln(x) - \int x \cdot \frac{1}{x}\,dx = x\ln(x) - \int 1\,dx$$

Step 3: Integrate.

$$= x\ln(x) - x + C$$

$$\boxed{\int \ln(x)\,dx = x\ln(x) - x + C}$$

Example 3 (Repeated IBP): Find $\displaystyle\int x^2 e^x \, dx$.

Step 1: LIPET gives $u = x^2$. Apply IBP.

$u = x^2$	$dv = e^x \, dx$
$du = 2x \, dx$	$v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \int x e^x \, dx$$

Step 2: The remaining integral $\int x e^x \, dx$ still requires IBP. Apply it again.

$u = x$	$dv = e^x \, dx$
$du = dx$	$v = e^x$

$$\int x e^x \, dx = x e^x - e^x$$

Step 3: Substitute back.

$$\int x^2 e^x \, dx = x^2 e^x - 2(x e^x - e^x) + C = x^2 e^x - 2x e^x + 2e^x + C$$

$$\boxed{\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C}$$

Example 4 (The Loop Trick): Find $\displaystyle\int e^x \sin(x) \, dx$.

Step 1: Let $I = \int e^x \sin(x) \, dx$. LIPET: e^x is exponential (E), $\sin(x)$ is trig (T). Since E comes before T in LIPET, choose $u = e^x$.

$u = e^x$	$dv = \sin(x) \, dx$
$du = e^x \, dx$	$v = -\cos(x)$

$$I = -e^x \cos(x) - \int (-\cos(x)) e^x \, dx = -e^x \cos(x) + \int e^x \cos(x) \, dx$$

Step 2: Apply IBP again to $\int e^x \cos(x) \, dx$. Keep $u = e^x$ (same choice as before).

$u = e^x$	$dv = \cos(x)\,dx$
$du = e^x\,dx$	$v = \sin(x)$

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \int e^x \sin(x)\,dx = e^x \sin(x) - I$$

Step 3: Substitute back into the expression for I.

$$I = -e^x \cos(x) + e^x \sin(x) - I$$

Step 4: Solve for I by adding I to both sides.

$$2I = e^x \sin(x) - e^x \cos(x)$$

$$I = \frac{e^x \sin(x) - e^x \cos(x)}{2} + C$$

$$\boxed{\int e^x \sin(x)\,dx = \frac{e^x \sin(x) - e^x \cos(x)}{2} + C}$$

Example 5 (Definite Integral with IBP): Evaluate $\displaystyle\int_1^e x \ln(x)\,dx$.

Step 1: LIPET gives $u = \ln(x)$ (L comes before P).

$u = \ln(x)$	$dv = x\,dx$
$du = \frac{1}{x}\,dx$	$v = \frac{x^2}{2}$

Step 2: Apply the formula with limits.

$$\int_1^e x \ln(x)\,dx = \left[\frac{x^2}{2}\ln(x)\right]_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \left[\frac{x^2}{2}\ln(x)\right]_1^e - \frac{1}{2}\int_1^e x\,dx$$

Step 3: Evaluate $\left[\frac{x^2}{2}\ln(x)\right]_1^e$.

$$\frac{e^2}{2}\ln(e) - \frac{1}{2}\ln(1) = \frac{e^2}{2}(1) - \frac{1}{2}(0) = \frac{e^2}{2}$$

Step 4: Evaluate $\frac{1}{2}\displaystyle\int_1^e x\,dx$.

$$\frac{1}{2}\left[\frac{x^2}{2}\right]_1^e = \frac{1}{2} \cdot \frac{e^2 - 1}{2} = \frac{e^2 - 1}{4}$$

Step 5: Subtract.

$$\frac{e^2}{2} - \frac{e^2 - 1}{4} = \frac{2e^2}{4} - \frac{e^2 - 1}{4} = \frac{2e^2 - e^2 + 1}{4} = \frac{e^2 + 1}{4}$$

$$\boxed{\int_1^e x \ln(x)\,dx = \frac{e^2 + 1}{4}}$$

Common Mistakes

- **Choosing u and dv backwards:** If after applying IBP the new integral is more complicated than the original, you likely have the wrong choice of u. Swap them and try again. LIPET helps prevent this.
- **Forgetting to differentiate u or integrate dv:** In the setup table, du is found by differentiating u, and v is found by integrating dv. Mixing these up (differentiating dv or integrating u) produces the wrong formula.
- **Switching the choice of u on the second IBP in the loop trick:** If you swap which factor is u on the second application, the two integrals cancel and give $0 = 0$. Always keep the *same type* of function as u in both applications.
- **Forgetting the negative sign from v:** When $dv = \sin(x)\,dx$, we get $v = -\cos(x)$. Dropping the negative sign here causes a sign error that propagates through the entire problem.
- **Adding $+C$ too early in repeated IBP:** Only add $+C$ at the very end after all integrations are complete and substituted back.

Worksheet

1. Find $\int xe^{2x}\,dx$.

2. Find $\int x\cos(x)\,dx$.

3. Find $\int x^2 \cos(x)\, dx$.

   (Requires two applications of IBP.)

4. Find $\int \ln(3x)\, dx$.

   (Set $dv = dx$.)

5. Find $\int x^2 \ln(x)\, dx$.

6. Find $\int \arctan(x)\, dx$.

   (Set $dv = dx$.)

7. Find $\displaystyle\int e^x \cos(x)\, dx$.

   (Use the loop trick.)

8. Evaluate $\displaystyle\int_0^1 xe^x\, dx$.

9. Evaluate $\displaystyle\int_1^e \ln(x)\, dx$.

10. (Challenge) Find $\displaystyle\int x^3 e^x\, dx$.

    (Requires three applications of IBP.)

11. (Challenge) Find $\int \sin(\ln(x))\,dx$.

Hint: Let $u = \sin(\ln(x))$ *and* $dv = dx$. *After two applications of IBP, use the loop trick.*

Answer Key

1. $\displaystyle\int xe^{2x}\,dx$

$u = x,\ dv = e^{2x}\,dx,\ du = dx,\ v = \dfrac{e^{2x}}{2}.$

$$= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$$\boxed{\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C}$$

2. $\displaystyle\int x\cos(x)\,dx$

$u = x,\ dv = \cos(x)\,dx,\ du = dx,\ v = \sin(x).$

$$= x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C$$

$$\boxed{\int x\cos(x)\,dx = x\sin(x) + \cos(x) + C}$$

3. $\displaystyle\int x^2\cos(x)\,dx$

First IBP: $u = x^2,\ dv = \cos(x)\,dx,\ du = 2x\,dx,\ v = \sin(x).$

$$= x^2\sin(x) - \int 2x\sin(x)\,dx = x^2\sin(x) - 2\int x\sin(x)\,dx$$

Second IBP on $\int x\sin(x)\,dx$: $u = x,\ dv = \sin(x)\,dx,\ du = dx,\ v = -\cos(x).$

$$\int x\sin(x)\,dx = -x\cos(x) - \int(-\cos(x))\,dx = -x\cos(x) + \sin(x)$$

Substitute back:

$$= x^2\sin(x) - 2(-x\cos(x) + \sin(x)) + C = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

$$\boxed{\int x^2\cos(x)\,dx = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C}$$

4. $\displaystyle\int \ln(3x)\,dx$

$u = \ln(3x),\ dv = dx,\ du = \dfrac{1}{x}\,dx,\ v = x.$

$$= x\ln(3x) - \int x\cdot\frac{1}{x}\,dx = x\ln(3x) - \int 1\,dx = x\ln(3x) - x + C$$

$$\boxed{\int \ln(3x)\,dx = x\ln(3x) - x + C}$$

5. $\int x^2 \ln(x)\,dx$

$u = \ln(x)$, $dv = x^2\,dx$, $du = \frac{1}{x}\,dx$, $v = \frac{x^3}{3}$.

$$= \frac{x^3}{3}\ln(x) - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\ln(x) - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C$$

$$\boxed{\int x^2\ln(x)\,dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C}$$

6. $\int \arctan(x)\,dx$

$u = \arctan(x)$, $dv = dx$, $du = \frac{1}{1+x^2}\,dx$, $v = x$.

$$= x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

For $\int \frac{x}{1+x^2}\,dx$, use u-sub with $w = 1+x^2$, $dw = 2x\,dx$:

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\ln|1+x^2| = \frac{1}{2}\ln(1+x^2)$$

$$\boxed{\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C}$$

7. $\int e^x\cos(x)\,dx$

Let $I = \int e^x \cos(x)\,dx$.

First IBP: $u = e^x$, $dv = \cos(x)\,dx$, $du = e^x\,dx$, $v = \sin(x)$.

$$I = e^x\sin(x) - \int e^x\sin(x)\,dx$$

Second IBP on $\int e^x\sin(x)\,dx$: $u = e^x$, $dv = \sin(x)\,dx$, $du = e^x\,dx$, $v = -\cos(x)$.

$$\int e^x\sin(x)\,dx = -e^x\cos(x) + \int e^x\cos(x)\,dx = -e^x\cos(x) + I$$

Substitute back:

$$I = e^x\sin(x) - (-e^x\cos(x) + I) = e^x\sin(x) + e^x\cos(x) - I$$

Solve for I:

$$2I = e^x \sin(x) + e^x \cos(x) \implies I = \frac{e^x \sin(x) + e^x \cos(x)}{2} + C$$

$$\boxed{\int e^x \cos(x)\, dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + C}$$

8. $\displaystyle\int_0^1 xe^x\, dx$
$u = x$, $dv = e^x\, dx$, $du = dx$, $v = e^x$.

$$= \Big[xe^x\Big]_0^1 - \int_0^1 e^x\, dx = \Big[xe^x\Big]_0^1 - \Big[e^x\Big]_0^1$$

$$= (1 \cdot e^1 - 0) - (e^1 - e^0) = e - e + 1 = 1$$

$$\boxed{\int_0^1 xe^x\, dx = 1}$$

9. $\displaystyle\int_1^e \ln(x)\, dx$
$u = \ln(x)$, $dv = dx$, $du = \dfrac{1}{x}\, dx$, $v = x$.

$$= \Big[x\ln(x)\Big]_1^e - \int_1^e 1\, dx = \Big[x\ln(x)\Big]_1^e - \Big[x\Big]_1^e$$

$$= (e\ln(e) - 1 \cdot \ln(1)) - (e - 1) = (e - 0) - (e - 1) = e - e + 1 = 1$$

$$\boxed{\int_1^e \ln(x)\, dx = 1}$$

10. (Challenge) $\displaystyle\int x^3 e^x\, dx$
Apply IBP three times, always choosing $u =$ the polynomial.
First IBP: $u = x^3$, $dv = e^x\, dx$, $du = 3x^2\, dx$, $v = e^x$.

$$= x^3 e^x - 3\int x^2 e^x\, dx$$

Second IBP on $\int x^2 e^x\, dx$: $u = x^2$, $dv = e^x\, dx$, $du = 2x\, dx$, $v = e^x$.

$$\int x^2 e^x\, dx = x^2 e^x - 2\int xe^x\, dx$$

Third IBP on $\int xe^x\, dx$: $u = x$, $dv = e^x\, dx$, $du = dx$, $v = e^x$.

$$\int xe^x\, dx = xe^x - e^x$$

Substitute back step by step:

$$\int x^2e^x\,dx = x^2e^x - 2(xe^x - e^x) = x^2e^x - 2xe^x + 2e^x$$

$$\int x^3e^x\,dx = x^3e^x - 3(x^2e^x - 2xe^x + 2e^x) + C$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

$$\boxed{\int x^3e^x\,dx = e^x(x^3 - 3x^2 + 6x - 6) + C}$$

11. (Challenge) $\displaystyle\int \sin(\ln(x))\,dx$

Let $I = \displaystyle\int \sin(\ln(x))\,dx$.

First IBP: $u = \sin(\ln(x))$, $dv = dx$, $du = \dfrac{\cos(\ln(x))}{x}\,dx$, $v = x$.

$$I = x\sin(\ln(x)) - \int x \cdot \frac{\cos(\ln(x))}{x}\,dx = x\sin(\ln(x)) - \int \cos(\ln(x))\,dx$$

Second IBP on $\int \cos(\ln(x))\,dx$: $u = \cos(\ln(x))$, $dv = dx$, $du = -\dfrac{\sin(\ln(x))}{x}\,dx$, $v = x$.

$$\int \cos(\ln(x))\,dx = x\cos(\ln(x)) - \int x \cdot \left(-\frac{\sin(\ln(x))}{x}\right)dx = x\cos(\ln(x)) + \int \sin(\ln(x))\,dx$$

$$= x\cos(\ln(x)) + I$$

Substitute back:

$$I = x\sin(\ln(x)) - (x\cos(\ln(x)) + I) = x\sin(\ln(x)) - x\cos(\ln(x)) - I$$

Solve for I:

$$2I = x\sin(\ln(x)) - x\cos(\ln(x))$$

$$\boxed{\int \sin(\ln(x))\,dx = \frac{x\sin(\ln(x)) - x\cos(\ln(x))}{2} + C}$$

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