Introduction to Differential Equations
ApexMath

Notes

What Is a Differential Equation?

Every equation you have solved so far has asked you to find a *number*. For example, solving $2x + 3 = 7$ gives you $x = 2$.

A **differential equation** is different. It is an equation that involves a function *and* its derivative. When you solve a differential equation, you are not looking for a number — you are looking for a **function**.

Here are some examples of differential equations:

$$\frac{dy}{dx} = 3x^2 \qquad \frac{dy}{dx} = 2y \qquad \frac{d^2y}{dx^2} + y = 0$$

Each of these equations contains a derivative. The goal is to find the function y that makes the equation true.

Why Do They Matter?

Differential equations describe how things *change*. In the real world:

- The rate at which a population grows depends on how large the population currently is.
- The rate at which a hot object cools depends on how much hotter it is than the room.
- The rate at which a drug leaves the body depends on how much of it remains.

In each case, the relationship between a quantity and its rate of change is a differential equation.

Solutions to Differential Equations

A **solution** to a differential equation is any function $y = f(x)$ that makes the equation true when substituted in.

To **verify** a solution, you differentiate it and substitute both y and $\frac{dy}{dx}$ (or higher derivatives if needed) back into the equation. If both sides are equal, it is a solution.

Example of verification: Is $y = e^{3x}$ a solution to $\frac{dy}{dx} = 3y$?

Step 1: Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 3e^{3x}$$

Step 2: Substitute into the right side of the equation.

$$3y = 3e^{3x}$$

Step 3: Compare both sides.

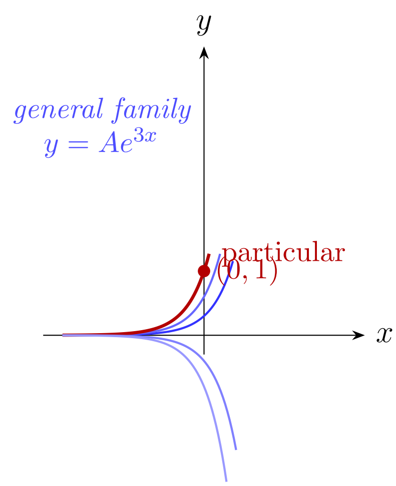
$$\frac{dy}{dx} = 3e^{3x} \quad \text{and} \quad 3y = 3e^{3x} \quad \checkmark$$

Both sides are equal, so $y = e^{3x}$ is a solution.

General Solutions and Particular Solutions

Most differential equations do not have just one solution. They have a whole *family* of solutions that differ by a constant C.

For example, every function of the form $y = e^{3x+C}$, or equivalently $y = Ae^{3x}$ (where A is any constant), is a solution to $\frac{dy}{dx} = 3y$. This is called the **general solution**.



When you are given an **initial condition** — a specific point (x_0, y_0) that the solution must pass through — you can find the exact value of C. The result is called a **particular solution**.

For example, if $\frac{dy}{dx} = 3y$ and we require $y(0) = 1$:

$$y = Ae^{3x} \quad \Rightarrow \quad 1 = Ae^0 = A \quad \Rightarrow \quad A = 1$$

The particular solution is $y = e^{3x}$.

Order of a Differential Equation

The **order** of a differential equation is the highest derivative that appears in it.

- $\frac{dy}{dx} = 2y$ is **first order** (highest derivative is y').
- $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$ is **second order** (highest derivative is y'').

In this unit, every differential equation you encounter will be first order.

Pro Tip

- **Verifying a solution is not the same as finding one.** To verify, you substitute and check. No solving is required. Go step by step: differentiate first, then substitute, then compare.

- **A general solution contains an arbitrary constant C.** If your answer has no C, you likely have a particular solution. If the problem did not give you an initial condition, your answer should still have C.

- **An initial condition is written as $y(x_0) = y_0$.** This means "the function equals y_0 when $x = x_0$." Substitute $x = x_0$ and $y = y_0$ into the general solution, then solve for C.

- **Plug C back in at the end.** After finding C, write out the full particular solution explicitly. Do not leave your answer as "$C = 2$."

- **Check your particular solution.** After finding it, substitute the initial condition back in to confirm it satisfies the equation.

Examples

Example 1 (Verifying a Solution): Verify that $y = x^3 + 2x$ is a solution to $\frac{dy}{dx} = 3x^2 + 2$.

Step 1: Differentiate y.

$$\frac{dy}{dx} = 3x^2 + 2$$

Step 2: Compare to the right side of the differential equation.
The right side is $3x^2 + 2$.

$$\frac{dy}{dx} = 3x^2 + 2 \quad \checkmark$$

Both sides match, so $y = x^3 + 2x$ is a solution.

Example 2 (Verifying a Solution — Substitution Required): Verify that $y = Ce^{-2x}$ is a solution to $\frac{dy}{dx} + 2y = 0$ for any constant C.

Step 1: Differentiate y.

$$\frac{dy}{dx} = -2Ce^{-2x}$$

Step 2: Substitute $\frac{dy}{dx}$ and y into the left side of the equation.

$$\frac{dy}{dx} + 2y = -2Ce^{-2x} + 2(Ce^{-2x}) = -2Ce^{-2x} + 2Ce^{-2x} = 0 \quad \checkmark$$

The left side equals 0, which matches the right side. So $y = Ce^{-2x}$ is a solution for any value of C.

Example 3 (Finding a Particular Solution): The general solution to a differential equation is $y = Ce^{x^2}$. Find the particular solution satisfying $y(0) = 5$.

Step 1: Substitute the initial condition $x = 0$, $y = 5$ into the general solution.

$$5 = Ce^{0^2} = Ce^0 = C \cdot 1 = C$$

Step 2: State C.

$$C = 5$$

Step 3: Write the particular solution.

$$\boxed{y = 5e^{x^2}}$$

Example 4 (Finding a Particular Solution — Solving for C): The general solution to a differential equation is $y = \dfrac{1}{x^2 + C}$. Find the particular solution satisfying $y(1) = 2$.

Step 1: Substitute $x = 1$, $y = 2$.

$$2 = \frac{1}{1^2 + C} = \frac{1}{1 + C}$$

Step 2: Solve for C.

$$2(1 + C) = 1$$

$$2 + 2C = 1$$

$$2C = -1$$

$$C = -\frac{1}{2}$$

Step 3: Write the particular solution.

$$\boxed{y = \frac{1}{x^2 - \frac{1}{2}} = \frac{2}{2x^2 - 1}}$$

Example 5 (Verifying and Finding Particular Solution): Given the differential equation $\dfrac{dy}{dx} = -y$ and general solution $y = Ce^{-x}$:

(a) Verify that $y = Ce^{-x}$ is a solution.

(b) Find the particular solution satisfying $y(0) = 4$.

Part (a):
Step 1: Differentiate $y = Ce^{-x}$.

$$\frac{dy}{dx} = -Ce^{-x}$$

Step 2: Compare to the right side $-y = -Ce^{-x}$.

$$\frac{dy}{dx} = -Ce^{-x} = -y \quad \checkmark$$

Part (b):
Step 1: Substitute $x = 0$, $y = 4$.

$$4 = Ce^0 = C$$

Step 2: Write the particular solution.

$$\boxed{y = 4e^{-x}}$$

Common Mistakes

- **Differentiating instead of substituting when verifying:** To verify a solution you must differentiate it, then substitute *both* y and $\frac{dy}{dx}$ into the original equation and check that both sides are equal. Students often differentiate but forget to substitute back.
- **Forgetting C in the general solution:** If you integrate to find a general solution and omit the constant C, you have already assumed a particular solution without justification.
- **Leaving the answer as $C = \ldots$:** After using the initial condition to find C, always substitute it back to write the complete particular solution.
- **Applying the initial condition to $\frac{dy}{dx}$ instead of y:** An initial condition $y(x_0) = y_0$ tells you the value of the *function*, not the derivative. Substitute into y, not $\frac{dy}{dx}$.
- **Mixing up order and degree:** The *order* is determined by the highest derivative present, not by the power of y or the number of terms.

Worksheet

1. Verify that $y = 2x^2 - x$ is a solution to $\dfrac{dy}{dx} = 4x - 1$.

2. Verify that $y = 5e^{4x}$ is a solution to $\dfrac{dy}{dx} = 4y$.

3. Verify that $y = Ce^{3x}$ is a solution to $\dfrac{dy}{dx} - 3y = 0$ for any constant C.

4. The general solution to a differential equation is $y = x^2 + C$. Find the particular solution satisfying $y(2) = 7$.

5. The general solution to a differential equation is $y = Ce^{5x}$. Find the particular solution satisfying $y(0) = -3$.

6. The general solution to a differential equation is $y = \dfrac{C}{x}$. Find the particular solution satisfying $y(4) = 2$.

7. The general solution to a differential equation is $y = \ln(x + C)$. Find the particular

solution satisfying $y(0) = 0$.

8. (Challenge) Verify that $y = x^2 + \frac{C}{x}$ is a solution to $x\frac{dy}{dx} + y = 3x^2$, then find the particular solution satisfying $y(1) = 4$.

Answer Key

1. Verify $y = 2x^2 - x$ is a solution to $\dfrac{dy}{dx} = 4x - 1$.

Step 1: Differentiate y.

$$\frac{dy}{dx} = 4x - 1$$

Step 2: Compare to the right side.
The right side is $4x - 1$.

$$\frac{dy}{dx} = 4x - 1 \quad \checkmark$$

Both sides match. $y = 2x^2 - x$ is a solution.

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2. Verify $y = 5e^{4x}$ is a solution to $\dfrac{dy}{dx} = 4y$.

Step 1: Differentiate y.

$$\frac{dy}{dx} = 5 \cdot 4e^{4x} = 20e^{4x}$$

Step 2: Compute $4y$.

$$4y = 4(5e^{4x}) = 20e^{4x}$$

Step 3: Compare.

$$\frac{dy}{dx} = 20e^{4x} = 4y \quad \checkmark$$

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3. Verify $y = Ce^{3x}$ is a solution to $\dfrac{dy}{dx} - 3y = 0$.

Step 1: Differentiate y.

$$\frac{dy}{dx} = 3Ce^{3x}$$

Step 2: Substitute into the left side.

$$\frac{dy}{dx} - 3y = 3Ce^{3x} - 3(Ce^{3x}) = 3Ce^{3x} - 3Ce^{3x} = 0 \quad \checkmark$$

The left side equals 0 for any value of C.

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4. General solution $y = x^2 + C$. Initial condition $y(2) = 7$.

Step 1: Substitute $x = 2$, $y = 7$.

$$7 = (2)^2 + C = 4 + C$$

Step 2: Solve for C.

$$C = 7 - 4 = 3$$

Step 3: Write the particular solution.

$$\boxed{y = x^2 + 3}$$

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5. General solution $y = Ce^{5x}$. Initial condition $y(0) = -3$.

Step 1: Substitute $x = 0$, $y = -3$.

$$-3 = Ce^0 = C \cdot 1 = C$$

Step 2: Write the particular solution.

$$\boxed{y = -3e^{5x}}$$

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6. General solution $y = \dfrac{C}{x}$. Initial condition $y(4) = 2$.

Step 1: Substitute $x = 4$, $y = 2$.

$$2 = \frac{C}{4}$$

Step 2: Solve for C.

$$C = 2 \times 4 = 8$$

Step 3: Write the particular solution.

$$\boxed{y = \frac{8}{x}}$$

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7. General solution $y = \ln(x + C)$. Initial condition $y(0) = 0$.

Step 1: Substitute $x = 0$, $y = 0$.

$$0 = \ln(0 + C) = \ln C$$

Step 2: Solve for C.
Recall that $\ln C = 0$ means $C = e^0 = 1$.

$$C = 1$$

Step 3: Write the particular solution.

$$\boxed{y = \ln(x + 1)}$$

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8. (Challenge) Verify $y = x^2 + \frac{C}{x}$ is a solution to $x\frac{dy}{dx} + y = 3x^2$, then find the particular solution for $y(1) = 4$.

Verification:
Step 1: Differentiate $y = x^2 + Cx^{-1}$.

$$\frac{dy}{dx} = 2x - Cx^{-2} = 2x - \frac{C}{x^2}$$

Step 2: Substitute into the left side $x\frac{dy}{dx} + y$.

$$x\left(2x - \frac{C}{x^2}\right) + \left(x^2 + \frac{C}{x}\right) = 2x^2 - \frac{C}{x} + x^2 + \frac{C}{x} = 3x^2 \quad \checkmark$$

Particular solution:
Step 3: Substitute $x = 1$, $y = 4$.

$$4 = (1)^2 + \frac{C}{1} = 1 + C$$

Step 4: Solve for C.

$$C = 3$$

Step 5: Write the particular solution.

$$\boxed{y = x^2 + \frac{3}{x}}$$

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