Lesson 2: Limit Laws and Properties

ApexMath

1 Notes on Limits

- Limits obey several algebraic laws that make evaluation easier.
- Basic limit laws:

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$
$$\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)$$
$$\lim_{x\to a}[f(x)\cdot g(x)] = (\lim_{x\to a} f(x))\cdot(\lim_{x\to a} g(x))$$
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \lim_{x\to a} g(x)\neq 0$$

- Power and root laws:

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n, \quad \lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

- Use these laws after checking if direct substitution works.
- Combine with factoring, standard limits, and hole analysis for more complicated expressions.

2 Deep Walkthrough Examples

Example A. Evaluate $\lim_{x\to 2}(3x^2+5x-1)$

Step 1. Linear/polynomial function, continuous everywhere

Step 2. Substitute $x=2$:

$$3(2)^2+5(2)-1$$

Step 3. Simplify:

$$12+10-1=21$$

$$\boxed{21}$$

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Example B. Evaluate $\lim_{x\to 1}\frac{x^2-1}{x-1}+2x$

Step 1. Split using limit laws:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}+\lim_{x\to 1}2x$$

Step 2. Factor numerator for first term:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1, \quad x \neq 1$$

Step 3. Apply substitution:

$$\lim_{x\to 1}(x+1) + \lim_{x\to 1} 2x = (1+1) + (2\cdot 1)$$

Step 4. Simplify:

$$2+2=4$$

$$\boxed{4}$$

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Example C. Evaluate $\lim_{x\to 0} \frac{x^2 \cos x}{\sin x}$

Step 1. Split using limit laws:

$$\lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \frac{\cos x}{\sin x}$$

Step 2. Rewrite fraction to use standard limit:

$$\frac{x^2 \cos x}{\sin x} = x \cdot \frac{x}{\sin x} \cdot \cos x$$

Step 3. Recognize $\frac{x}{\sin x} \to 1$ as standard:

$$x \cdot \frac{x}{\sin x} \to 0 \cdot 1 = 0$$

Step 4. Multiply by $\lim_{x\to 0} \cos x = 1$

$$\boxed{0}$$

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3 Common Mistakes

- Forgetting to check if direct substitution works first.
- Not factoring properly when a removable discontinuity exists.
- Misapplying limit laws to non-continuous or undefined points.
- Confusing standard limits (like $\frac{\sin x}{x}$) and forgetting to apply them correctly.
- Arithmetic mistakes when simplifying after substitution.

4 Worksheet Problems

1. $\lim_{x\to 3}(2x^2 - x + 4)$
2. $\lim_{x\to 2} \frac{x^2-4}{x-2} + 5$
3. $\lim_{x\to 0} x \cdot \frac{\sin x}{x}$
4. $\lim_{x\to 1} \frac{x^3-1}{x-1} - 2$
5. $\lim_{x\to -1}(x^2 + 3x + 2) + \frac{x^2-1}{x+1}$

5 Answer Key (Step-by-Step)

1. $\lim_{x\to 3}(2x^2 - x + 4)$

Step 1. Polynomial, continuous

Step 2. Substitute $x = 3$:

$$2(3)^2 - 3 + 4$$

Step 3. Simplify:

$$18 - 3 + 4 = 19$$

$$\boxed{19}$$

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2. $\lim_{x\to 2} \frac{x^2-4}{x-2} + 5$

Step 1. Split using limit law:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} + \lim_{x\to 2} 5$$

Step 2. Factor numerator:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2, \quad x \neq 2$$

Step 3. Substitute $x = 2$:

$$(2+2) + 5 = 4 + 5$$

Step 4. Simplify:

$$9$$

$$\boxed{9}$$

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3. $\lim_{x\to 0} x \cdot \frac{\sin x}{x}$

Step 1. Split using limit law:

$$\lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{\sin x}{x}$$

Step 2. Apply standard limit: $\frac{\sin x}{x} \to 1$

Step 3. Multiply by $\lim_{x\to 0} x = 0$

$$\boxed{0}$$

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4. $\lim_{x\to 1} \frac{x^3-1}{x-1} - 2$

Step 1. Factor numerator (difference of cubes):

$$x^3 - 1 = (x-1)(x^2+x+1)$$

Step 2. Simplify fraction:

$$\frac{(x-1)(x^2+x+1)}{x-1} = x^2 + x + 1, \quad x \neq 1$$

Step 3. Substitute $x = 1$:

$$1 + 1 + 1 - 2 = 1$$

$$\boxed{1}$$

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5. $\lim_{x\to -1}(x^2 + 3x + 2) + \frac{x^2-1}{x+1}$

Step 1. Factor numerator of second term:

$$x^2 - 1 = (x-1)(x+1)$$

Step 2. Simplify fraction:

$$\frac{(x-1)(x+1)}{x+1} = x - 1, \quad x \neq -1$$

Step 3. Substitute $x = -1$ in first and simplified second term:

$$((-1)^2 + 3(-1) + 2) + (-1-1) = (1 - 3 + 2) + (-2) = 0 - 2 = -2$$

$$\boxed{-2}$$

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