Lesson 8: Comprehensive Limits Review: Techniques, Continuity, and Applications

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Notes on Limits

- A **limit** describes the value a function approaches as the input approaches a certain point.
- **Standard limit forms** include $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 1^\infty, 0^0, \infty^0$.
- **One-sided limits** evaluate the approach from the left ($x \to a^-$) or right ($x \to a^+$).
- **Two-sided limits** exist only if both one-sided limits are equal.
- **Limits at infinity** determine the behavior of a function as $x \to \infty$ or $x \to -\infty$.
- **Continuity**: A function $f(x)$ is continuous at $x = a$ if
  - $f(a)$ is defined,
  - $\lim_{x\to a} f(x)$ exists,
  - $\lim_{x\to a} f(x) = f(a)$
- **Intermediate Value Theorem (IVT)**: If f is continuous on $[a, b]$ and k is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = k$.
- **Connection to tangent lines**: The derivative $f'(a)$ is the limit of the difference quotient, linking instantaneous slope with limits.

Pro Tips

- Always check for indeterminate forms before attempting to plug in values.
- Factor or rationalize to simplify expressions and eliminate $0/0$ situations.
- Remember the difference between one-sided and two-sided limits—failure to check sides can cause mistakes.
- For limits at infinity, divide by the highest power of x in the denominator or numerator to simplify.
- Continuous functions on closed intervals satisfy the IVT—use this when finding roots or values in context.
- **Put it all together now:** identify the type of limit, simplify algebraically, check one- and two-sided limits, then interpret the result.

Deep Walkthrough Examples

Example A. Evaluate $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

Step 1: Factor numerator:

$$x^2 - 4 = (x - 2)(x + 2)$$

Step 2: Cancel $x - 2$:

$$\frac{(x-2)(x+2)}{x-2} = x + 2$$

Step 3: Plug in $x = 2$:

$$2 + 2 = 4$$

$$\boxed{\lim_{x\to 2} \frac{x^2-4}{x-2} = 4}$$

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Example B. Evaluate $\lim_{x\to 0^+} \frac{1}{x}$.

- As x approaches 0 from the right, the values of $1/x$ become very large positive numbers.

$$\boxed{\lim_{x\to 0^+} \frac{1}{x} = +\infty}$$

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Example C. Check continuity of $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$.

Step 1: Factor numerator:

$$x^2 - 1 = (x-1)(x+1)$$

Step 2: Simplify:

$$f(x) = x + 1 \text{ for } x \neq 1$$

Step 3: Evaluate $\lim_{x\to 1} f(x) = 2$.

Step 4: $f(1)$ is undefined in original function, so **f** is not continuous at $x = 1$.

$$\boxed{\text{Discontinuous at } x = 1}$$

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Common Mistakes

- Forgetting to simplify before plugging in leads to 0/0 errors.
- Confusing one-sided limits with two-sided limits.
- Assuming continuity without checking definition.
- Misinterpreting limits at infinity—remember it's about behavior, not plugging in infinity.
- Ignoring the domain of the function.

Worksheet Problems

1. Evaluate $\lim_{x\to 3} \frac{x^2-9}{x-3}$.

2. Evaluate $\lim_{x\to 0^-} \frac{1}{x}$.

3. Evaluate $\lim_{x\to\infty} \frac{3x^2+2}{5x^2-x}$.

4. Determine if $f(x) = \frac{x^2-1}{x-1}$ is continuous at $x = 1$.
5. Use IVT: $f(x) = x^3 - 2$ on $[1, 2]$. Show a c exists such that $f(c) = 0$.
6. Evaluate $\lim_{x\to 1} \frac{x^3-1}{x-1}$.
7. Connect limit to tangent line: Find slope of tangent to $f(x) = x^2$ at $x = 3$.

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Detailed Answer Key

1. $\lim_{x\to 3} \frac{x^2-9}{x-3}$

$$x^2 - 9 = (x-3)(x+3)$$

$$\frac{(x-3)(x+3)}{x-3} = x + 3$$

$$\lim_{x\to 3}(x+3) = 6$$

$$\boxed{6}$$

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2. $\lim_{x\to 0^-} \frac{1}{x}$

As x approaches 0 from the left, $1/x \to -\infty$.

$$\boxed{\text{-}\infty}$$

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3. $\lim_{x\to\infty} \frac{3x^2+2}{5x^2-x}$

Divide numerator and denominator by x^2:

$$\frac{3 + 2/x^2}{5 - 1/x} \to \frac{3}{5}$$

$$\boxed{3/5}$$

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4. $f(x) = \frac{x^2-1}{x-1}$ at $x = 1$

Factor and simplify: $f(x) = x + 1$, but $f(1)$ undefined in original function.

$$\boxed{\text{Discontinuous at } x = 1}$$

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5. IVT for $f(x) = x^3 - 2$ on $[1, 2]$

$$f(1) = -1, \quad f(2) = 6$$

0 lies between -1 and 6, so by IVT, $\exists c \in (1, 2)$ such that $f(c) = 0$.

$$\boxed{c \in (1, 2)}$$

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6. $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Factor numerator: $x^3 - 1 = (x - 1)(x^2 + x + 1)$

Cancel $x - 1$: $\frac{(x-1)(x^2+x+1)}{x-1} = x^2 + x + 1$

Plug in $x = 1$: $1 + 1 + 1 = 3$

$$\boxed{3}$$

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7. Slope of tangent to $f(x) = x^2$ at $x = 3$

$$f'(x) = \lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} (6 + h) = 6$$

$$\boxed{6}$$

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