Defining the Definite Integral as a Limit of Riemann Sums
ApexMath

Notes

Where We Are Coming From

In the previous lesson we used rectangles to *approximate* the area under a curve. We found that the more rectangles we used, the closer our approximation got to the true area. This lesson takes that idea one step further: what happens when we let the number of rectangles grow to *infinity*?

The answer is that the approximation becomes exact, and the result is called the **definite integral**.

The Setup

Let $f(x)$ be a function defined on a closed interval $[a, b]$. Divide $[a, b]$ into n equal subintervals. Each subinterval has width:

$$\Delta x = \frac{b-a}{n}$$

The endpoints of the subintervals are:

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_n = b$$

In general, the i-th endpoint is:

$$x_i = a + i \cdot \Delta x = a + i \cdot \frac{b-a}{n}$$

The Riemann Sum

A **Riemann sum** is formed by choosing a sample point x_i^* in each subinterval, evaluating f there, multiplying by the width Δx, and summing all n rectangles:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

The most common choice of sample point is the **right endpoint**:

$$x_i^* = x_i = a + i \cdot \frac{b-a}{n}$$

So the right-endpoint Riemann sum is:

$$\sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n}$$

The Definition of the Definite Integral

As $n \to \infty$, the rectangles get infinitely thin and the sum converges to the exact area. This limit is the **definite integral**:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

This is not just a formula — it is the *definition* of what an integral is. Every time you write $\int_a^b f(x)\,dx$, this limit is what it means.

The Two Skills in This Lesson

This lesson focuses on moving fluently in **both directions**:

- **Integral → Riemann Sum:** Given a definite integral, rewrite it as an explicit limit of a sigma sum.
- **Riemann Sum → Integral:** Given a limit of a sigma sum, recognize and write the corresponding definite integral.

How to Go From an Integral to a Riemann Sum

Given $\displaystyle\int_a^b f(x)\,dx$, follow these steps:

Step 1: Write $\Delta x = \dfrac{b-a}{n}$.

Step 2: Write the right endpoint $x_i = a + i \cdot \dfrac{b-a}{n}$.

Step 3: Substitute into the sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{i(b-a)}{n}\right) \cdot \frac{b-a}{n}$$

Step 4: Simplify $f(x_i)$ by expanding the expression inside f.

How to Go From a Riemann Sum to an Integral

Given a limit like $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$, you need to identify:

- $\Delta x = \dfrac{b-a}{n}$: read off $b-a$ from the coefficient of $\dfrac{1}{n}$.
- $x_i^* = a + i\,\Delta x$: identify what is being multiplied by i.
- f: what function is being evaluated at x_i^*.
- a and b: recover the limits of integration.

Then write the integral $\int_a^b f(x)\,dx$.

Useful Summation Formulas
When evaluating the limit directly, these formulas are essential:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Taking the Limit: Key Technique
After applying the summation formulas, you will get an expression in n. To evaluate the limit as $n \to \infty$, divide every term by the highest power of n in the denominator. Any term with n remaining in the denominator goes to zero.

For example:

$$\lim_{n\to\infty} \frac{2n^2 + 3n + 1}{n^2} = \lim_{n\to\infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = 2$$

Pro Tip

- When converting an integral to a Riemann sum, always use right endpoints unless a problem specifically asks for left or midpoint. Right endpoints are the standard convention for this type of problem.
- After writing $x_i = a + \frac{i(b-a)}{n}$, immediately substitute it into $f(x)$ and simplify *before* writing the full sum. Trying to expand inside the sigma sign all at once leads to algebra errors.
- When going from a Riemann sum back to an integral, look at the term multiplied by i inside f. That term is Δx, and it tells you $b - a$. Then figure out a from the constant term.
- After applying summation formulas, always factor before taking the limit. Cancelling n terms early prevents messy algebra.
- Double-check your answer: if you compute the limit and it gives a reasonable finite number, that is a good sign. If you get ∞ or something undefined, go back and check your summation formula application.

Examples

Example 1 (Integral → Riemann Sum): Write $\int_0^3 (x^2+1)\,dx$ as a limit of a Riemann sum using right endpoints, then evaluate the limit.

Step 1: Identify a, b, and f.

$$a = 0, \quad b = 3, \quad f(x) = x^2 + 1$$

Step 2: Write Δx and x_i.

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}, \qquad x_i = 0 + i \cdot \frac{3}{n} = \frac{3i}{n}$$

Step 3: Evaluate $f(x_i)$.

$$f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^2 + 1 = \frac{9i^2}{n^2} + 1$$

Step 4: Write the Riemann sum.

$$\int_0^3 (x^2+1)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + 1\right) \cdot \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27i^2}{n^3} + \frac{3}{n}\right)$$

Step 5: Split the sum and apply summation formulas.

$$= \lim_{n\to\infty} \left[\frac{27}{n^3} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \cdot n\right]$$

$$= \lim_{n\to\infty} \left[\frac{27(n+1)(2n+1)}{6n^2} + 3\right]$$

Step 6: Take the limit. Divide numerator and denominator by n^2.

$$\lim_{n\to\infty} \frac{27(n+1)(2n+1)}{6n^2} = \frac{27}{6} \cdot \lim_{n\to\infty} \frac{2n^2+3n+1}{n^2} = \frac{27}{6} \cdot 2 = 9$$

$$\int_0^3 (x^2+1)\,dx = 9 + 3$$

$$\boxed{\int_0^3 (x^2+1)\,dx = 12}$$

Example 2 (Integral → Riemann Sum, non-zero lower limit): Write $\int_1^4 (3x-2)\,dx$ as a limit of a Riemann sum, then evaluate.

Step 1: Identify a, b, and f.

$$a = 1, \quad b = 4, \quad f(x) = 3x - 2$$

Step 2: Write Δx and x_i.

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}, \qquad x_i = 1 + i \cdot \frac{3}{n} = 1 + \frac{3i}{n}$$

Step 3: Evaluate $f(x_i)$.

$$f\left(1 + \frac{3i}{n}\right) = 3\left(1 + \frac{3i}{n}\right) - 2 = 3 + \frac{9i}{n} - 2 = 1 + \frac{9i}{n}$$

Step 4: Write the Riemann sum.

$$\int_1^4 (3x-2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{9i}{n}\right) \cdot \frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n} + \frac{27i}{n^2}\right)$$

Step 5: Split and apply summation formulas.

$$= \lim_{n\to\infty} \left[\frac{3}{n} \cdot n + \frac{27}{n^2} \cdot \frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \left[3 + \frac{27(n+1)}{2n}\right]$$

Step 6: Take the limit.

$$\lim_{n\to\infty} \frac{27(n+1)}{2n} = \frac{27}{2} \cdot \lim_{n\to\infty} \frac{n+1}{n} = \frac{27}{2} \cdot 1 = \frac{27}{2}$$

$$\int_1^4 (3x-2)\,dx = 3 + \frac{27}{2} = \frac{6}{2} + \frac{27}{2}$$

$$\boxed{\int_1^4 (3x-2)\,dx = \frac{33}{2}}$$

Example 3 (Riemann Sum → Integral): Identify the definite integral represented by:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{5i}{n}\right)^3 \cdot \frac{5}{n}$$

Step 1: Identify Δx.

The factor $\frac{5}{n}$ at the end is Δx, so:

$$\Delta x = \frac{b-a}{n} = \frac{5}{n} \implies b - a = 5$$

Step 2: Identify x_i.
The expression being cubed is $x_i^* = 2 + \frac{5i}{n}$. This matches $a + i\,\Delta x$, so:

$$a = 2, \qquad b = a + 5 = 7$$

Step 3: Identify f.
The function being evaluated at x_i^* is $f(x) = x^3$.
Step 4: Write the integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{5i}{n}\right)^3 \cdot \frac{5}{n}$$

$$\boxed{= \int_2^7 x^3\,dx}$$

Example 4 (Riemann Sum → Integral, trickier): Identify the definite integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi i}{n}\right) \cdot \frac{\pi}{n}$$

Step 1: Identify Δx.

$$\Delta x = \frac{\pi}{n} \implies b - a = \pi$$

Step 2: Identify x_i.
The expression inside sin is $\frac{\pi i}{n} = i \cdot \Delta x$. This matches $a + i\,\Delta x$ with $a = 0$:

$$a = 0, \qquad b = 0 + \pi = \pi$$

Step 3: Identify f.

$$f(x) = \sin(x)$$

Step 4: Write the integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi i}{n}\right) \cdot \frac{\pi}{n}$$

$$\boxed{= \int_0^{\pi} \sin(x)\,dx}$$

Common Mistakes

- **Forgetting to substitute x_i into f before writing the sum:** You must replace x with $a + \dfrac{i(b-a)}{n}$ inside f. Writing $f(x) \cdot \dfrac{b-a}{n}$ with x still in it is not a valid Riemann sum — x must be replaced by an expression in i and n.

- **Confusing Δx with x_i:** $\Delta x = \dfrac{b-a}{n}$ is the width of one rectangle. $x_i = a + i\,\Delta x$ is the location of the right endpoint. These are two different things and play different roles.

- **Applying the wrong summation formula:** Check carefully whether you have $\sum i$, $\sum i^2$, or $\sum i^3$. Using the i^2 formula when you have i^3 is a common and costly error.

- **Forgetting the constant term when splitting sums:** If the sum has a term with no i in it, like $\dfrac{3}{n}$, apply $\sum_{i=1}^{n} 1 = n$ to handle it. Do not drop constant terms.

- **Misidentifying a when going from sum to integral:** The constant term inside $f(x_i^*)$ that does not involve i is a. Students sometimes treat the entire expression as x_i and miss the lower limit.

Worksheet

Part A: Integral → Riemann Sum (Write the limit, do not evaluate.)

1. Write $\displaystyle\int_0^5 (2x+3)\,dx$ as a limit of a Riemann sum using right endpoints.

2. Write $\displaystyle\int_0^2 x^3\,dx$ as a limit of a Riemann sum using right endpoints.

3. Write $\int_2^6 (x^2 - 4)\, dx$ as a limit of a Riemann sum using right endpoints.

4. Write $\int_1^3 \frac{1}{x}\, dx$ as a limit of a Riemann sum using right endpoints.

Part B: Integral → Riemann Sum, then Evaluate the Limit.

5. Write $\int_0^1 (4x + 1)\, dx$ as a Riemann sum and evaluate the limit.

$$\text{Recall: } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

6. Write $\int_0^3 x^2\, dx$ as a Riemann sum and evaluate the limit.

$$\text{Recall: } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

7. Write $\int_1^3 (x^2 + x)\,dx$ as a Riemann sum and evaluate the limit.

8. Write $\int_0^2 (x^3 - x)\,dx$ as a Riemann sum and evaluate the limit.

$$\text{Recall: } \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Part C: Riemann Sum → Integral (Identify the definite integral.)

9. $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n}\right)^2 \cdot \frac{4}{n}$

10. $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n}$

11. $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \left(3 + \frac{i}{n}\right)^4 \cdot \frac{1}{n}$

12. $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{2\pi i}{n}\right) \cdot \frac{2\pi}{n}$

13. (Challenge) $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + \frac{5i}{n}} \cdot \frac{5}{n}$

14. (Challenge) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e^{i/n}$

Answer Key

Part A

1. $\displaystyle\int_0^5 (2x+3)\,dx$

$a = 0,\ b = 5,\ \Delta x = \dfrac{5}{n},\ x_i = \dfrac{5i}{n}$

$f(x_i) = 2\left(\dfrac{5i}{n}\right) + 3 = \dfrac{10i}{n} + 3$

$$\boxed{\int_0^5 (2x+3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{10i}{n} + 3\right) \cdot \frac{5}{n}}$$

2. $\displaystyle\int_0^2 x^3\,dx$

$a = 0,\ b = 2,\ \Delta x = \dfrac{2}{n},\ x_i = \dfrac{2i}{n}$

$f(x_i) = \left(\dfrac{2i}{n}\right)^3 = \dfrac{8i^3}{n^3}$

$$\boxed{\int_0^2 x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^3}{n^3} \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i^3}{n^4}}$$

3. $\displaystyle\int_2^6 (x^2-4)\,dx$

$a = 2,\ b = 6,\ \Delta x = \dfrac{4}{n},\ x_i = 2 + \dfrac{4i}{n}$

$f(x_i) = \left(2 + \dfrac{4i}{n}\right)^2 - 4 = 4 + \dfrac{16i}{n} + \dfrac{16i^2}{n^2} - 4 = \dfrac{16i}{n} + \dfrac{16i^2}{n^2}$

$$\boxed{\int_2^6 (x^2-4)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{16i}{n} + \frac{16i^2}{n^2}\right) \cdot \frac{4}{n}}$$

4. $\displaystyle\int_1^3 \frac{1}{x}\,dx$

$a = 1,\ b = 3,\ \Delta x = \dfrac{2}{n},\ x_i = 1 + \dfrac{2i}{n}$

$f(x_i) = \dfrac{1}{1 + \frac{2i}{n}} = \dfrac{n}{n+2i}$

$$\boxed{\int_1^3 \frac{1}{x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n+2i} \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n+2i}}$$

Part B

5. $\displaystyle\int_0^1 (4x+1)\,dx$

$\Delta x = \dfrac{1}{n},\ x_i = \dfrac{i}{n},\ f(x_i) = \dfrac{4i}{n} + 1$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n} + 1\right) \cdot \frac{1}{n} = \lim_{n\to\infty} \left[\frac{4}{n^2} \sum_{i=1}^{n} i + \frac{1}{n} \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{4}{n^2} \cdot \frac{n(n+1)}{2} + \frac{1}{n} \cdot n\right] = \lim_{n\to\infty} \left[\frac{2(n+1)}{n} + 1\right]$$

$$= \lim_{n\to\infty} \left[2 + \frac{2}{n} + 1\right] = 2 + 0 + 1$$

$$\boxed{\int_0^1 (4x+1)\,dx = 3}$$

6. $\displaystyle\int_0^3 x^2\,dx$

$\Delta x = \dfrac{3}{n},\ x_i = \dfrac{3i}{n},\ f(x_i) = \dfrac{9i^2}{n^2}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{9i^2}{n^2} \cdot \frac{3}{n} = \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{27(n+1)(2n+1)}{6n^2} = \frac{27}{6} \cdot \lim_{n\to\infty} \frac{2n^2+3n+1}{n^2} = \frac{27}{6} \cdot 2 = \frac{54}{6}$$

$$\boxed{\int_0^3 x^2\,dx = 9}$$

7. $\displaystyle\int_1^3 (x^2+x)\,dx$

$\Delta x = \dfrac{2}{n},\ x_i = 1 + \dfrac{2i}{n}$

$f(x_i) = \left(1 + \dfrac{2i}{n}\right)^2 + \left(1 + \dfrac{2i}{n}\right) = 1 + \dfrac{4i}{n} + \dfrac{4i^2}{n^2} + 1 + \dfrac{2i}{n} = 2 + \dfrac{6i}{n} + \dfrac{4i^2}{n^2}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{6i}{n} + \frac{4i^2}{n^2}\right) \cdot \frac{2}{n} = \lim_{n\to\infty} \left[\frac{4}{n} \cdot n + \frac{12}{n^2} \cdot \frac{n(n+1)}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[4 + \frac{6(n+1)}{n} + \frac{4(n+1)(2n+1)}{3n^2}\right]$$

$$= 4 + 6 + \frac{4 \cdot 2}{3} = 10 + \frac{8}{3} = \frac{30}{3} + \frac{8}{3}$$

$$\boxed{\int_1^3 (x^2 + x)\,dx = \frac{38}{3}}$$

8. $\int_0^2 (x^3 - x)\,dx$

$\Delta x = \frac{2}{n},\ x_i = \frac{2i}{n}$

$f(x_i) = \left(\frac{2i}{n}\right)^3 - \frac{2i}{n} = \frac{8i^3}{n^3} - \frac{2i}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i^3}{n^3} - \frac{2i}{n}\right) \cdot \frac{2}{n} = \lim_{n\to\infty} \left[\frac{16}{n^4} \cdot \left(\frac{n(n+1)}{2}\right)^2 - \frac{4}{n^2} \cdot \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{4(n+1)^2}{n^2} - \frac{2(n+1)}{n}\right] = \lim_{n\to\infty} \left[4 \cdot \frac{(n+1)^2}{n^2} - 2 \cdot \frac{n+1}{n}\right] = 4(1) - 2(1)$$

$$\boxed{\int_0^2 (x^3 - x)\,dx = 2}$$

Part C

9. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n}\right)^2 \cdot \frac{4}{n}$

$\Delta x = \frac{4}{n}$, so $b - a = 4$. The expression inside f is $x_i = \frac{4i}{n}$, giving $a = 0$, $b = 4$. Function: $f(x) = x^2$.

$$\boxed{= \int_0^4 x^2\,dx}$$

10. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n}$

$\Delta x = \frac{2}{n}$, so $b - a = 2$. The expression inside f is $x_i = 1 + \frac{2i}{n}$, giving $a = 1$, $b = 3$. Function: $f(x) = x$.

$$\boxed{= \int_1^3 x\,dx}$$

11. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(3 + \frac{i}{n}\right)^4 \cdot \frac{1}{n}$

$\Delta x = \frac{1}{n}$, so $b - a = 1$. The expression inside f is $x_i = 3 + \frac{i}{n}$, giving $a = 3$, $b = 4$. Function: $f(x) = x^4$.

$$\boxed{= \int_3^4 x^4\,dx}$$

12. $\lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{2\pi i}{n}\right) \cdot \frac{2\pi}{n}$

$\Delta x = \frac{2\pi}{n}$, so $b - a = 2\pi$. The expression inside cos is $x_i = \frac{2\pi i}{n}$, giving $a = 0$, $b = 2\pi$. Function: $f(x) = \cos(x)$.

$$\boxed{= \int_0^{2\pi} \cos(x)\, dx}$$

13. (Challenge) $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + \frac{5i}{n}} \cdot \frac{5}{n}$

$\Delta x = \frac{5}{n}$, so $b - a = 5$. The expression under the square root is $x_i = 4 + \frac{5i}{n}$, giving $a = 4$, $b = 9$. Function: $f(x) = \sqrt{x}$.

$$\boxed{= \int_4^{9} \sqrt{x}\, dx}$$

14. (Challenge) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e^{i/n}$

Rewrite to make Δx visible:

$$\frac{1}{n} \sum_{i=1}^{n} e^{i/n} = \sum_{i=1}^{n} e^{i/n} \cdot \frac{1}{n}$$

$\Delta x = \frac{1}{n}$, so $b - a = 1$. The exponent is $x_i = \frac{i}{n} = \frac{1 \cdot i}{n}$, giving $a = 0$, $b = 1$. Function: $f(x) = e^x$.

$$\boxed{= \int_0^{1} e^x\, dx}$$

Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

End of Lesson • ApexMath
