Linearization and Linear Approximation
ApexMath

Notes

What is Linearization?

When a function is complicated, we can use its **derivative** to build a simple straight line that closely matches the function near a specific point. This straight line is called the **linearization** of the function.

Think of it this way: if you zoom in far enough on any smooth curve, it starts to look like a straight line. Linearization takes advantage of that idea.

The Formula:

The linearization of a function $f(x)$ at a point $x = a$ is:

$$L(x) = f(a) + f'(a)(x - a)$$

This is the equation of the **tangent line** to $f(x)$ at $x = a$.

Where does this formula come from?

Recall the point-slope form of a line from algebra:

$$y - y_1 = m(x - x_1)$$

For linearization:

- The point on the curve is $(a,\ f(a))$, so $x_1 = a$ and $y_1 = f(a)$
- The slope of the tangent line at $x = a$ is $m = f'(a)$

Substituting these into point-slope form:

$$y - f(a) = f'(a)(x - a)$$

Solving for y:

$$y = f(a) + f'(a)(x - a)$$

This is exactly the linearization formula. We rename y as $L(x)$ to show it is a function that *approximates* $f(x)$.

How do we use it?

Once we have $L(x)$, we can plug in a value of x that is close to a to get an **approximate value** of $f(x)$ without computing the exact (often difficult) value.

$$f(x) \approx L(x) \quad \text{when } x \text{ is close to } a$$

Pro Tip

- The linearization formula is just the tangent line equation. If you remember how to find a tangent line, you already know how to do linearization.
- Always find $f(a)$ and $f'(a)$ *first*, before writing out $L(x)$. Doing them separately reduces mistakes.
- The closer x is to a, the more accurate the approximation will be.
- If you are asked to "approximate" a value using a derivative, linearization is almost always the right method.
- Double-check that you are evaluating both f and f' at the same point $x = a$.

Examples

Example 1: Find the linearization of $f(x) = \sqrt{x}$ at $x = 9$. Then use it to approximate $\sqrt{9.1}$.

Step 1: Identify the function and the center point.
We have $f(x) = \sqrt{x}$ and $a = 9$.

Step 2: Find $f(a)$.
Plug $x = 9$ into the original function:

$$f(9) = \sqrt{9} = 3$$

Step 3: Find the derivative $f'(x)$.
Rewrite the square root as a power to make differentiation easier:

$$f(x) = x^{1/2} \implies f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Step 4: Find $f'(a)$.
Plug $x = 9$ into the derivative:

$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$$

Step 5: Write the linearization formula.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Step 6: Use $L(x)$ to approximate $\sqrt{9.1}$.
Plug in $x = 9.1$:

$$L(9.1) = 3 + \frac{1}{6}(9.1 - 9) = 3 + \frac{1}{6}(0.1) = 3 + \frac{0.1}{6} \approx 3 + 0.0167$$

$$\boxed{\sqrt{9.1} \approx 3.0167}$$

(The actual value is ≈ 3.01662, so our approximation is very close!)

Example 2: Find the linearization of $f(x) = x^3$ at $x = 2$. Then approximate $(2.05)^3$.

Step 1: Identify the function and the center point.
We have $f(x) = x^3$ and $a = 2$.

Step 2: Find $f(a)$.

$$f(2) = 2^3 = 8$$

Step 3: Find the derivative $f'(x)$.

$$f'(x) = 3x^2$$

Step 4: Find $f'(a)$.

$$f'(2) = 3(2)^2 = 3 \cdot 4 = 12$$

Step 5: Write the linearization.

$$L(x) = f(a) + f'(a)(x - a) = 8 + 12(x - 2)$$

Step 6: Approximate $(2.05)^3$ by plugging in $x = 2.05$.

$$L(2.05) = 8 + 12(2.05 - 2) = 8 + 12(0.05) = 8 + 0.6 = 8.6$$

$$\boxed{(2.05)^3 \approx 8.6}$$

(The actual value is 8.615125, so our approximation is very reasonable.)

Example 3: Find the linearization of $f(x) = \sin(x)$ at $x = 0$. Then approximate $\sin(0.1)$.

Step 1: Identify the function and center point.
We have $f(x) = \sin(x)$ and $a = 0$.

Step 2: Find $f(a)$.

$$f(0) = \sin(0) = 0$$

Step 3: Find the derivative.

$$f'(x) = \cos(x)$$

Step 4: Find $f'(a)$.

$$f'(0) = \cos(0) = 1$$

Step 5: Write the linearization.

$$L(x) = 0 + 1(x - 0) = x$$

This tells us something powerful: near $x = 0$, $\sin(x) \approx x$. This is a very famous result in mathematics and physics!

Step 6: Approximate $\sin(0.1)$.

$$L(0.1) = 0.1$$

$$\boxed{\sin(0.1) \approx 0.1}$$

(The actual value is ≈ 0.09983, which is extremely close.)

Common Mistakes

- **Forgetting to evaluate f' at $x = a$:** Students sometimes write the general derivative formula instead of plugging in a. You must substitute a to get a number for the slope.
- **Mixing up $f(a)$ and $f'(a)$:** Make sure $f(a)$ comes from the original function, and $f'(a)$ comes from the derivative. Do not plug a into the wrong one.
- **Choosing a value of a far from x:** Linearization is only accurate when x is close to a. Do not use $a = 0$ to approximate something like $f(100)$.
- **Forgetting the $(x - a)$ term:** A common error is writing $L(x) = f(a) + f'(a) \cdot x$ instead of $f(a) + f'(a)(x - a)$. The shift $(x - a)$ is essential.
- **Not simplifying $f(a)$ or $f'(a)$ before substituting:** Always compute these values as clean numbers first. Carrying unsimplified expressions leads to errors later.

Worksheet

1. Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$. Then use it to approximate $\sqrt{4.2}$.

2. Find the linearization of $f(x) = x^2$ at $a = 3$. Then approximate $(3.1)^2$.

3. Find the linearization of $f(x) = \cos(x)$ at $a = 0$. Then approximate $\cos(0.05)$.

4. Find the linearization of $f(x) = e^x$ at $a = 0$. Then approximate $e^{0.2}$.

5. Find the linearization of $f(x) = \ln(x)$ at $a = 1$. Then approximate $\ln(1.3)$.

6. Find the linearization of $f(x) = x^4$ at $a = 1$. Then approximate $(1.02)^4$.

7. Find the linearization of $f(x) = \frac{1}{x}$ at $a = 2$. Then approximate $\frac{1}{2.1}$.

8. (Challenge) Find the linearization of $f(x) = \sqrt[3]{x}$ at $a = 8$. Then approximate $\sqrt[3]{8.4}$.

9. (Challenge) Explain in your own words why the linearization is more accurate when x is very close to a compared to when x is far from a.

10. (Challenge) A student says the linearization of $f(x) = x^2$ at $a = 5$ is $L(x) = 25 + 10x$. Identify the student's mistake and write the correct linearization.

Answer Key

1. Linearization of $f(x) = \sqrt{x}$ at $a = 4$. Approximate $\sqrt{4.2}$.

Step 1: Find $f(a)$.

$$f(4) = \sqrt{4} = 2$$

Step 2: Find the derivative.

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Step 3: Evaluate the derivative at $a = 4$.

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$$

Step 4: Write the linearization.

$$L(x) = 2 + \frac{1}{4}(x - 4)$$

Step 5: Approximate $\sqrt{4.2}$ by plugging in $x = 4.2$.

$$L(4.2) = 2 + \frac{1}{4}(4.2 - 4) = 2 + \frac{1}{4}(0.2) = 2 + 0.05$$

$$\boxed{\sqrt{4.2} \approx 2.05}$$

2. Linearization of $f(x) = x^2$ at $a = 3$. Approximate $(3.1)^2$.

Step 1: Find $f(a)$.

$$f(3) = 3^2 = 9$$

Step 2: Find the derivative.

$$f'(x) = 2x$$

Step 3: Evaluate at $a = 3$.

$$f'(3) = 2(3) = 6$$

Step 4: Write the linearization.

$$L(x) = 9 + 6(x - 3)$$

Step 5: Plug in $x = 3.1$.

$$L(3.1) = 9 + 6(3.1 - 3) = 9 + 6(0.1) = 9 + 0.6$$

$$\boxed{(3.1)^2 \approx 9.6}$$

3. Linearization of $f(x) = \cos(x)$ at $a = 0$. Approximate $\cos(0.05)$.

Step 1: Find $f(a)$.

$$f(0) = \cos(0) = 1$$

Step 2: Find the derivative.

$$f'(x) = -\sin(x)$$

Step 3: Evaluate at $a = 0$.

$$f'(0) = -\sin(0) = 0$$

Step 4: Write the linearization.

$$L(x) = 1 + 0 \cdot (x - 0) = 1$$

Step 5: Plug in $x = 0.05$.

$$L(0.05) = 1$$

$$\boxed{\cos(0.05) \approx 1}$$

(Note: The actual value is ≈ 0.99875. The approximation is reasonable for small x.)

4. Linearization of $f(x) = e^x$ at $a = 0$. Approximate $e^{0.2}$.

Step 1: Find $f(a)$.

$$f(0) = e^0 = 1$$

Step 2: Find the derivative. (Recall: the derivative of e^x is e^x.)

$$f'(x) = e^x$$

Step 3: Evaluate at $a = 0$.

$$f'(0) = e^0 = 1$$

Step 4: Write the linearization.

$$L(x) = 1 + 1 \cdot (x - 0) = 1 + x$$

Step 5: Plug in $x = 0.2$.

$$L(0.2) = 1 + 0.2$$

$$\boxed{e^{0.2} \approx 1.2}$$

5. Linearization of $f(x) = \ln(x)$ at $a = 1$. Approximate $\ln(1.3)$.

Step 1: Find $f(a)$.

$$f(1) = \ln(1) = 0$$

Step 2: Find the derivative.

$$f'(x) = \frac{1}{x}$$

Step 3: Evaluate at $a = 1$.

$$f'(1) = \frac{1}{1} = 1$$

Step 4: Write the linearization.

$$L(x) = 0 + 1 \cdot (x - 1) = x - 1$$

Step 5: Plug in $x = 1.3$.

$$L(1.3) = 1.3 - 1 = 0.3$$

$$\boxed{\ln(1.3) \approx 0.3}$$

6. Linearization of $f(x) = x^4$ at $a = 1$. Approximate $(1.02)^4$.

Step 1: Find $f(a)$.

$$f(1) = 1^4 = 1$$

Step 2: Find the derivative.

$$f'(x) = 4x^3$$

Step 3: Evaluate at $a = 1$.

$$f'(1) = 4(1)^3 = 4$$

Step 4: Write the linearization.

$$L(x) = 1 + 4(x - 1)$$

Step 5: Plug in $x = 1.02$.

$$L(1.02) = 1 + 4(1.02 - 1) = 1 + 4(0.02) = 1 + 0.08$$

$$\boxed{(1.02)^4 \approx 1.08}$$

7. Linearization of $f(x) = \dfrac{1}{x}$ at $a = 2$. Approximate $\dfrac{1}{2.1}$.

Step 1: Find $f(a)$.

$$f(2) = \frac{1}{2}$$

Step 2: Find the derivative. Rewrite as $f(x) = x^{-1}$, then differentiate.

$$f'(x) = -x^{-2} = -\frac{1}{x^2}$$

Step 3: Evaluate at $a = 2$.

$$f'(2) = -\frac{1}{(2)^2} = -\frac{1}{4}$$

Step 4: Write the linearization.

$$L(x) = \frac{1}{2} + \left(-\frac{1}{4}\right)(x - 2) = \frac{1}{2} - \frac{1}{4}(x - 2)$$

Step 5: Plug in $x = 2.1$.

$$L(2.1) = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.5 - \frac{1}{4}(0.1) = 0.5 - 0.025$$

$$\boxed{\frac{1}{2.1} \approx 0.475}$$

8. (Challenge) Linearization of $f(x) = \sqrt[3]{x}$ at $a = 8$. Approximate $\sqrt[3]{8.4}$.

Step 1: Rewrite the function using exponent notation.

$$f(x) = x^{1/3}$$

Step 2: Find $f(a)$.

$$f(8) = 8^{1/3} = 2$$

Step 3: Find the derivative.

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

Step 4: Evaluate at $a = 8$.

$$f'(8) = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$$

Step 5: Write the linearization.

$$L(x) = 2 + \frac{1}{12}(x - 8)$$

Step 6: Plug in $x = 8.4$.

$$L(8.4) = 2 + \frac{1}{12}(8.4 - 8) = 2 + \frac{0.4}{12} = 2 + 0.0\overline{3} \approx 2 + 0.0333$$

$$\boxed{\sqrt[3]{8.4} \approx 2.033}$$

9. (Challenge) Why is linearization more accurate close to a?

Step 1: Understand what the tangent line represents.

The linearization $L(x)$ is the tangent line at $x = a$. It touches the curve *exactly* at the point $(a, f(a))$ and matches the slope of the curve at that point.

Step 2: Think about what happens as we move away from a.

As x moves further from a, the curve $f(x)$ starts to bend — but the line $L(x)$ stays straight. The line cannot follow the curve's bending, so the gap between $L(x)$ and $f(x)$ grows.

Step 3: Conclusion.

The closer x is to a, the less the curve has had a chance to bend away from the tangent line, so the approximation is more accurate.

10. (Challenge) Identify the error and correct the linearization of $f(x) = x^2$ at $a = 5$.

Step 1: Find $f(a)$ and $f'(a)$.

$$f(5) = 25, \quad f'(x) = 2x, \quad f'(5) = 10$$

Step 2: Identify the student's error.

The student wrote $L(x) = 25 + 10x$. This is incorrect because they used $10x$ instead of $10(x - 5)$. They forgot to subtract $a = 5$ inside the parentheses.

Step 3: Write the correct linearization.

$$L(x) = 25 + 10(x - 5)$$

This can also be simplified:

$$L(x) = 25 + 10x - 50 = 10x - 25$$

$$\boxed{L(x) = 10x - 25}$$

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