Logistic Models
ApexMath

AP Calculus BC Only

> **Note for Students:** This topic is tested on the **AP Calculus BC** exam only. If you are in AP Calculus AB, you do not need to learn this material. BC students: logistic models appear frequently on the free-response section and are worth studying carefully.

Notes

The Problem with Exponential Growth

In the Exponential Models lesson, you solved $\dfrac{dP}{dt} = kP$, which produces unlimited growth. In the real world, however, populations are limited by food, space, and resources. Growth slows as a population approaches the maximum size the environment can support.

The Logistic Differential Equation

The **logistic model** modifies the exponential model by including a **carrying capacity** M — the maximum sustainable population. The model is:

$$\frac{dP}{dt} = kP(M - P)$$

Here is the intuition behind each part:

- When P is small compared to M, the factor $(M - P) \approx M$ is large, so the population grows nearly exponentially.
- As P approaches M, the factor $(M - P)$ approaches 0, slowing growth.
- If $P = M$, then $\dfrac{dP}{dt} = 0$ — the population stops changing.
- If $P > M$ (overpopulation), then $(M - P) < 0$, so $\dfrac{dP}{dt} < 0$ and the population declines back toward M.

The Solution Formula

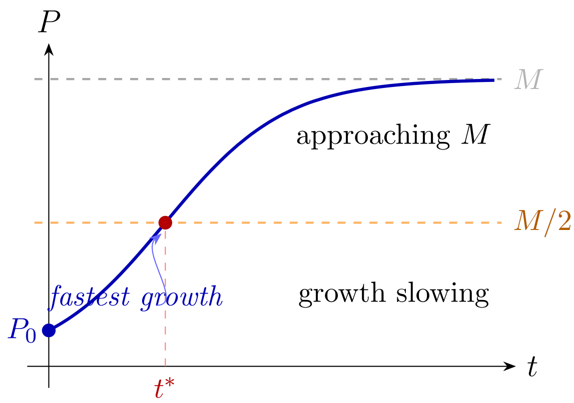
The logistic equation is separable. After separating and using partial fractions to integrate the left side, the solution is:

$$\boxed{P(t) = \frac{M}{1 + Ae^{-kMt}}}$$

where $A = \dfrac{M - P_0}{P_0}$ and $P_0 = P(0)$ is the initial population.

You do not need to re-derive this from scratch on the exam. Memorize the formula and the expression for A.

Behavior of the Logistic Curve



Key features of the logistic curve:

- The curve is **S-shaped** (sigmoidal).
- Growth is fastest at the **inflection point**, which occurs when $P = \dfrac{M}{2}$.
- The curve has **horizontal asymptotes** at $P = 0$ (unstable) and $P = M$ (stable). Every solution with $P_0 > 0$ approaches M as $t \to \infty$.

Finding the Inflection Point

At the inflection point, $\dfrac{d^2P}{dt^2} = 0$. It can be shown that this occurs exactly when $P = \dfrac{M}{2}$.

To find the *time* t^* at which the inflection point occurs, set $P(t^*) = \dfrac{M}{2}$ in the solution formula and solve for t^*:

$$\frac{M}{2} = \frac{M}{1 + Ae^{-kMt^*}} \implies 1 + Ae^{-kMt^*} = 2 \implies Ae^{-kMt^*} = 1 \implies t^* = \frac{\ln A}{kM}$$

Pro Tip

- **Memorize $A = \dfrac{M - P_0}{P_0}$ and the solution formula.** Every logistic problem starts by computing A. Get comfortable with this step.

- **Identify kM as a single quantity.** In the formula $P(t) = \dfrac{M}{1 + Ae^{-kMt}}$, the exponent contains kM together. You often do not need k and M separately — just kM.

- **The inflection point is always at $P = M/2$.** If asked "when is growth fastest?", the answer is always when P is exactly half the carrying capacity. Find the corresponding time by solving $P(t^*) = M/2$.

- **As $t \to \infty$, $P \to M$.** The carrying capacity is the long-run limit. Any answer that has the population exceeding M permanently is wrong.

- **Read the differential equation carefully.** The logistic DE is $\dfrac{dP}{dt} = kP(M - P)$. Sometimes it is written as $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right)$. These are equivalent — factor out M from the second form if needed.

Examples

Example 1 (Setting Up and Solving): A population satisfies $\dfrac{dP}{dt} = 0.001\,P(100 - P)$ with $P(0) = 10$.

(a) Write the solution $P(t)$.
(b) Find $P(5)$.
(c) Find $\lim_{t\to\infty} P(t)$.

Part (a): Identify $k = 0.001$, $M = 100$, $P_0 = 10$.

$$A = \frac{M - P_0}{P_0} = \frac{100 - 10}{10} = 9$$

$$kM = 0.001 \times 100 = 0.1$$

$$P(t) = \frac{100}{1 + 9e^{-0.1t}}$$

Part (b):

$$P(5) = \frac{100}{1 + 9e^{-0.5}} = \frac{100}{1 + 9(0.6065)} \approx \frac{100}{6.4587}$$

$$\boxed{P(5) \approx 15.48}$$

Part (c): As $t \to \infty$, the term $e^{-0.1t} \to 0$, so:

$$\lim_{t\to\infty} P(t) = \frac{100}{1 + 0} = 100 = M$$

$$\boxed{\lim_{t\to\infty} P(t) = 100}$$

Example 2 (Finding When P Reaches a Given Value): A population satisfies $\dfrac{dP}{dt} = 0.002\,P(500 - P)$ with $P(0) = 50$. Find when $P = 250$.

Step 1: Find A and write $P(t)$.

$$A = \frac{500 - 50}{50} = 9, \qquad kM = 0.002 \times 500 = 1$$

$$P(t) = \frac{500}{1 + 9e^{-t}}$$

Step 2: Set $P(t) = 250$ and solve for t.

$$250 = \frac{500}{1 + 9e^{-t}} \implies 1 + 9e^{-t} = 2 \implies 9e^{-t} = 1 \implies e^{-t} = \frac{1}{9} \implies t = \ln 9$$

$$\boxed{t = \ln 9 \approx 2.20}$$

Note: $P = 250 = M/2$, so this is the inflection point — the time of fastest growth.

Example 3 (Reading the Formula): A population follows $P(t) = \dfrac{200}{1 + 7e^{-0.3t}}$.

(a) Identify the carrying capacity M and initial population P_0.
(b) Find when $P = 100$.

Part (a):
The carrying capacity is the numerator: $M = 200$.
The initial population is found by setting $t = 0$:

$$P(0) = \frac{200}{1 + 7e^0} = \frac{200}{1 + 7} = \frac{200}{8} = 25$$

Part (b): $P = 100 = M/2$, so this is the inflection point.

$$100 = \frac{200}{1 + 7e^{-0.3t}} \implies 1 + 7e^{-0.3t} = 2 \implies 7e^{-0.3t} = 1$$

$$e^{-0.3t} = \frac{1}{7} \implies -0.3t = -\ln 7 \implies t = \frac{\ln 7}{0.3}$$

$$\boxed{t = \frac{\ln 7}{0.3} \approx 6.49}$$

Example 4 (Qualitative Analysis): For the logistic equation $\dfrac{dP}{dt} = kP(M - P)$ with $k > 0$:

(a) For what values of P is the population increasing? Decreasing?
(b) Where is the growth rate greatest?

(c) What are the equilibrium solutions?

Part (a):

$\frac{dP}{dt} = kP(M - P) > 0$ when both factors P and $(M - P)$ have the same sign. Since $k > 0$ and $P \geq 0$:

$$\text{Increasing when } 0 < P < M; \qquad \text{Decreasing when } P > M.$$

Part (b):

The growth rate $\frac{dP}{dt} = kP(M-P)$ is a quadratic in P (opening downward). Its maximum occurs at the vertex:

$$P = \frac{M}{2}$$

Growth is fastest when the population is at half the carrying capacity.

Part (c):

Equilibria occur when $\frac{dP}{dt} = 0$:

$$kP(M - P) = 0 \implies P = 0 \quad \text{or} \quad P = M$$

$P = 0$ is an **unstable** equilibrium (any small positive population grows). $P = M$ is a **stable** equilibrium (populations near M return to M).

Common Mistakes

- **Forgetting that the formula uses kM, not just k:** The exponent in the solution is $-kMt$. If you use $-kt$ instead, your answer will be wrong. Always compute kM first and use it as a single quantity.

- **Computing A incorrectly:** The formula is $A = \frac{M - P_0}{P_0}$. A common error is writing $A = \frac{P_0}{M - P_0}$ (the reciprocal). Double-check by confirming $P(0) = \frac{M}{1 + A} = P_0$.

- **Saying the population reaches M at some finite time:** The logistic curve has $P = M$ as a horizontal asymptote. The population approaches M but technically never equals it at any finite t.

- **Forgetting to check whether you need k alone or kM:** In most logistic problems, kM appears as a unit. If given separate values of k and M, multiply them first before substituting.

- **Confusing the inflection point $P = M/2$ with the time t^*:** The inflection point is a *population value*, not a time. To find the *time* of fastest growth, you must solve $P(t^*) = M/2$ for t^*.

Worksheet

1. A population satisfies $\dfrac{dP}{dt} = 0.002\,P(200 - P)$ with $P(0) = 20$.
   (a) Write the solution $P(t)$.
   (b) Find $P(3)$.
   (c) Find $\lim_{t \to \infty} P(t)$.

2. The population of a fish pond is modeled by $P(t) = \dfrac{500}{1 + 4e^{-0.5t}}$.
   (a) Identify the carrying capacity M and the initial population P_0.
   (b) Identify the constant k.
   (c) Find $P(4)$.

3. Using the model from problem 2, find the time t^* when the population reaches half the carrying capacity. This is the time of fastest growth.

4. A population satisfies $\frac{dP}{dt} = 0.001\,P(1000 - P)$ with $P(0) = 100$.

   (a) Write the solution $P(t)$.

   (b) Find $P(2)$.

   (c) Find the time of fastest growth.

5. A population follows $\frac{dP}{dt} = 0.004\,P(50 - P)$ with $P(0) = 5$. Find the time when $P = 40$.

6. A population is modeled by $P(t) = \dfrac{300}{1 + 14e^{-0.6t}}$.

   (a) Find M and P_0.

   (b) Find the time of the inflection point (fastest growth).

7. For the logistic equation $\dfrac{dP}{dt} = 0.005\,P(400 - P)$, answer the following without solving the equation:

   (a) What is the carrying capacity?

   (b) For what values of P is $\dfrac{dP}{dt}$ positive? Negative?

   (c) At what value of P is the growth rate greatest?

   (d) What are the two equilibrium solutions?

8. (Challenge) A logistic population has carrying capacity $M = 800$. You are told that $P(0) = 80$ and $P(5) = 400$. Find the value of k.

Answer Key

1. $\dfrac{dP}{dt} = 0.002\,P(200 - P)$, $P(0) = 20$.

(a) Find A and kM:

$$A = \frac{200 - 20}{20} = 9, \qquad kM = 0.002 \times 200 = 0.4$$

$$\boxed{P(t) = \frac{200}{1 + 9e^{-0.4t}}}$$

(b)

$$P(3) = \frac{200}{1 + 9e^{-1.2}} \approx \frac{200}{1 + 9(0.3012)} \approx \frac{200}{3.7108}$$

$$\boxed{P(3) \approx 53.90}$$

(c)

$$\lim_{t \to \infty} P(t) = 200 = M$$

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2. $P(t) = \dfrac{500}{1 + 4e^{-0.5t}}$.

(a) The numerator gives $M = 500$. Initial population:

$$P(0) = \frac{500}{1 + 4} = \frac{500}{5} = 100$$

$$\boxed{M = 500, \quad P_0 = 100}$$

(b) From $A = \dfrac{M - P_0}{P_0} = \dfrac{400}{100} = 4$ (confirmed) and $kM = 0.5$:

$$k = \frac{0.5}{500} = 0.001$$

$$\boxed{k = 0.001}$$

(c)

$$P(4) = \frac{500}{1 + 4e^{-2}} \approx \frac{500}{1 + 4(0.1353)} \approx \frac{500}{1.5413}$$

$$\boxed{P(4) \approx 324.39}$$

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3. Find t^* when $P = 250 = M/2$ for $P(t) = \dfrac{500}{1 + 4e^{-0.5t}}$.

Set $P(t^*) = 250$:

$$250 = \frac{500}{1 + 4e^{-0.5t^*}} \implies 1 + 4e^{-0.5t^*} = 2 \implies 4e^{-0.5t^*} = 1 \implies e^{-0.5t^*} = \frac{1}{4}$$

$$-0.5t^* = -\ln 4 \implies t^* = \frac{\ln 4}{0.5} = 2\ln 4$$

$$\boxed{t^* = 2\ln 4 \approx 2.77}$$

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4. $\dfrac{dP}{dt} = 0.001\,P(1000 - P)$, $P(0) = 100$.

(a) $A = \dfrac{1000 - 100}{100} = 9$, $kM = 0.001 \times 1000 = 1$.

$$\boxed{P(t) = \frac{1000}{1 + 9e^{-t}}}$$

(b)

$$P(2) = \frac{1000}{1 + 9e^{-2}} \approx \frac{1000}{1 + 9(0.1353)} \approx \frac{1000}{2.2180}$$

$$\boxed{P(2) \approx 450.85}$$

(c) Fastest growth at $P = M/2 = 500$. Set $P(t^*) = 500$:

$$500 = \frac{1000}{1 + 9e^{-t^*}} \implies 1 + 9e^{-t^*} = 2 \implies 9e^{-t^*} = 1 \implies t^* = \ln 9$$

$$\boxed{t^* = \ln 9 \approx 2.20}$$

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5. $\dfrac{dP}{dt} = 0.004\,P(50 - P)$, $P(0) = 5$. Find when $P = 40$.

Step 1: Find A and kM, write $P(t)$.

$$A = \frac{50 - 5}{5} = 9, \qquad kM = 0.004 \times 50 = 0.2$$

$$P(t) = \frac{50}{1 + 9e^{-0.2t}}$$

Step 2: Set $P(t) = 40$ and solve.

$$40 = \frac{50}{1 + 9e^{-0.2t}} \implies 1 + 9e^{-0.2t} = \frac{50}{40} = \frac{5}{4} \implies 9e^{-0.2t} = \frac{1}{4}$$

$$e^{-0.2t} = \frac{1}{36} \implies -0.2t = -\ln 36 \implies t = \frac{\ln 36}{0.2} = 5\ln 36$$

$$\boxed{t = 5\ln 36 \approx 17.92}$$

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6. $P(t) = \dfrac{300}{1 + 14e^{-0.6t}}$.

(a)

$$M = 300, \qquad P_0 = \frac{300}{1+14} = \frac{300}{15} = 20$$

$$\boxed{M = 300, \quad P_0 = 20}$$

(b) Inflection at $P = M/2 = 150$:

$$150 = \frac{300}{1 + 14e^{-0.6t^*}} \implies 1 + 14e^{-0.6t^*} = 2 \implies 14e^{-0.6t^*} = 1 \implies e^{-0.6t^*} = \frac{1}{14}$$

$$t^* = \frac{\ln 14}{0.6}$$

$$\boxed{t^* = \frac{\ln 14}{0.6} \approx 4.40}$$

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7. $\dfrac{dP}{dt} = 0.005\,P(400 - P)$.

(a) Carrying capacity: $M = 400$.

(b) $\dfrac{dP}{dt} > 0$ when $0 < P < 400$ (population is increasing).

$\dfrac{dP}{dt} < 0$ when $P > 400$ (population is decreasing).

(c) Maximum growth rate at $P = \dfrac{400}{2} = 200$.

(d) Equilibria at $P = 0$ (unstable) and $P = 400$ (stable).

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8. (Challenge) $M = 800$, $P(0) = 80$, $P(5) = 400$. Find k.

Step 1: Find A.

$$A = \frac{800 - 80}{80} = \frac{720}{80} = 9$$

Step 2: Write $P(t)$ in terms of k.

$$P(t) = \frac{800}{1 + 9e^{-800kt}}$$

Step 3: Apply $P(5) = 400 = M/2$.

$$400 = \frac{800}{1 + 9e^{-4000k}} \implies 1 + 9e^{-4000k} = 2 \implies 9e^{-4000k} = 1$$

$$e^{-4000k} = \frac{1}{9} \implies -4000k = -\ln 9 \implies k = \frac{\ln 9}{4000}$$

$$\boxed{k = \frac{\ln 9}{4000} \approx 0.000549}$$

$$\text{Verify: } P(5) = \frac{800}{1 + 9e^{-800 \cdot \ln 9/4000 \cdot 5}} = \frac{800}{1 + 9e^{-\ln 9}} = \frac{800}{1 + 9 \cdot \frac{1}{9}} = \frac{800}{2} = 400. \quad \checkmark$$

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