# Newton's Method
# ApexMath

## Notes

**What is Newton's Method?**

Sometimes we need to find where a function equals zero — that is, we need to solve $f(x) = 0$. These solutions are called **roots** or **zeros** of the function. For simple functions this is easy, but for complicated ones it can be nearly impossible to solve by hand.

Newton's Method is a technique that uses **derivatives** to make repeated, improving guesses at the root. Each guess gets closer and closer to the true answer.

**The Big Idea:**

Start with an initial guess $x_0$ near the root. Draw the tangent line to the curve at $x_0$. The tangent line crosses the $x$-axis at a new point, $x_1$. That new point is a *better* guess. Repeat the process from $x_1$ to get $x_2$, then from $x_2$ to get $x_3$, and so on.

Each new value is called an **iteration**.

**The Formula:**

To go from one guess to the next, we use:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In plain words: the next guess equals the current guess, minus the function value divided by the derivative value at that guess.

**Where does this formula come from?**

At the current guess $x_n$, we know the point on the curve is $\big(x_n,\ f(x_n)\big)$ and the slope of the tangent line is $f'(x_n)$.

The tangent line through this point is:

$$y - f(x_n) = f'(x_n)(x - x_n)$$

We want to find where this tangent line hits the $x$-axis, meaning $y = 0$:

$$0 - f(x_n) = f'(x_n)(x - x_n)$$

$$-f(x_n) = f'(x_n)(x - x_n)$$

Divide both sides by $f'(x_n)$ (assuming it is not zero):

$$\frac{-f(x_n)}{f'(x_n)} = x - x_n$$

Solve for $x$:

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

This new $x$ value is our next guess, $x_{n+1}$. That is exactly the Newton's Method formula.

**What if no starting guess is given?**

Sometimes a problem will simply say "use Newton's Method to approximate a root" without telling you what $x_0$ to use. In that case, you need to **choose your own starting guess**. Here is how to do it reliably:

- **Test a few simple integers.** Plug in $x = 0, 1, 2, -1, -2, \ldots$ into $f(x)$ and look at the signs of the outputs.
- **Find a sign change.** If $f(a) < 0$ and $f(b) > 0$ for two values $a$ and $b$, then by the Intermediate Value Theorem a root must exist somewhere between $a$ and $b$.
- **Pick the endpoint closer to zero.** Choose whichever of $a$ or $b$ has an $f$-value closer to zero as your $x_0$. That is the better starting point.

For example, suppose $f(1) = -2$ and $f(2) = 3$. There is a root between 1 and 2. Since $|-2|$ and $|3|$ are both reasonable, either works — but $x_0 = 1$ or $x_0 = 2$ are both good choices. Avoid starting far away from the sign change.

**When do we stop?**

We stop when the answer has reached enough decimal places of accuracy for our purposes. Typically, we either:

- Perform a fixed number of iterations (e.g. three iterations), or
- Stop when two consecutive guesses agree to several decimal places.

# Pro Tip

- Before starting, always find $f'(x)$ *once* and write it down. You will be plugging numbers into it repeatedly, so having it ready saves time and prevents errors.
- Organize your work in a table with columns for $n$, $x_n$, $f(x_n)$, $f'(x_n)$, and $x_{n+1}$. This keeps the repetitive steps clean and easy to check.
- Your initial guess $x_0$ should be reasonably close to the root. If no guess is given, test a few integers, find where $f(x)$ changes sign, and start at the endpoint whose $f$-value is closer to zero.
- When picking your own $x_0$, always verify the sign change first. Do not just guess randomly — a bad starting point can send the iterations in the wrong direction entirely.
- Newton's Method can fail if $f'(x_n) = 0$ at any step (division by zero). If this happens, choose a different starting guess.
- The method converges very quickly — typically each iteration roughly doubles the number of correct decimal places.

## Examples

**Example 1:** Use Newton's Method with $x_0 = 2$ to approximate a root of $f(x) = x^2 - 5$. Perform three iterations.

**Step 1: Identify the function and find the derivative.**

$$f(x) = x^2 - 5 \qquad f'(x) = 2x$$

*Note: The root of $f(x) = x^2 - 5$ is $x = \sqrt{5} \approx 2.2360679...$ so starting at $x_0 = 2$ is a reasonable guess.*

**Step 2: Write the Newton's Method formula for this function.**

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}$$

**Step 3: Perform Iteration 1 (find $x_1$ from $x_0 = 2$).**
Plug $x_0 = 2$ into $f$ and $f'$:

$$f(2) = (2)^2 - 5 = 4 - 5 = -1$$

$$f'(2) = 2(2) = 4$$

Apply the formula:

$$x_1 = 2 - \frac{-1}{4} = 2 + 0.25 = 2.25$$

**Step 4: Perform Iteration 2 (find $x_2$ from $x_1 = 2.25$).**

$$f(2.25) = (2.25)^2 - 5 = 5.0625 - 5 = 0.0625$$

$$f'(2.25) = 2(2.25) = 4.5$$

$$x_2 = 2.25 - \frac{0.0625}{4.5} \approx 2.25 - 0.013889 \approx 2.236111$$

**Step 5: Perform Iteration 3 (find $x_3$ from $x_2 \approx 2.236111$).**

$$f(2.236111) = (2.236111)^2 - 5 \approx 5.000193 - 5 = 0.000193$$

$$f'(2.236111) = 2(2.236111) \approx 4.472222$$

$$x_3 \approx 2.236111 - \frac{0.000193}{4.472222} \approx 2.236111 - 0.0000432 \approx 2.236068$$

$$\boxed{\sqrt{5} \approx 2.236068}$$

(The actual value is 2.2360679... — our answer is accurate to 6 decimal places after only 3 iterations!)

**Example 2:** Use Newton's Method with $x_0 = 1$ to approximate a root of $f(x) = x^3 - x - 1$. Perform three iterations.

**Step 1: Find the derivative.**

$$f(x) = x^3 - x - 1 \qquad f'(x) = 3x^2 - 1$$

**Step 2: Write the Newton's Method formula for this function.**

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$$

**Step 3: Iteration 1 from $x_0 = 1$.**

$$f(1) = (1)^3 - (1) - 1 = 1 - 1 - 1 = -1$$

$$f'(1) = 3(1)^2 - 1 = 3 - 1 = 2$$

$$x_1 = 1 - \frac{-1}{2} = 1 + 0.5 = 1.5$$

**Step 4: Iteration 2 from $x_1 = 1.5$.**

$$f(1.5) = (1.5)^3 - (1.5) - 1 = 3.375 - 1.5 - 1 = 0.875$$

$$f'(1.5) = 3(1.5)^2 - 1 = 3(2.25) - 1 = 6.75 - 1 = 5.75$$

$$x_2 = 1.5 - \frac{0.875}{5.75} \approx 1.5 - 0.152174 \approx 1.347826$$

**Step 5: Iteration 3 from $x_2 \approx 1.347826$.**

$$f(1.347826) \approx (1.347826)^3 - (1.347826) - 1 \approx 2.44554 - 1.347826 - 1 \approx 0.097714$$

$$f'(1.347826) = 3(1.347826)^2 - 1 \approx 3(1.81664) - 1 \approx 5.44992 - 1 = 4.44992$$

$$x_3 \approx 1.347826 - \frac{0.097714}{4.44992} \approx 1.347826 - 0.021960 \approx 1.325866$$

$$\boxed{x \approx 1.3259}$$

**Example 3 (Choose your own $x_0$):** Use Newton's Method to approximate a root of $f(x) = x^3 - x - 3$. No starting guess is given. Choose one, then perform three iterations.

**Step 1: Find the derivative.**

$$f(x) = x^3 - x - 3 \qquad f'(x) = 3x^2 - 1$$

**Step 2: Choose a starting guess by testing simple integers.**
Plug in a few values to find a sign change:

$$f(1) = (1)^3 - (1) - 3 = 1 - 1 - 3 = -3 \quad \text{(negative)}$$

$$f(2) = (2)^3 - (2) - 3 = 8 - 2 - 3 = 3 \quad \text{(positive)}$$

There is a sign change between $x = 1$ and $x = 2$, so a root exists in that interval. Both endpoints are close to zero, so we pick $x_0 = 1$ since $|f(1)| = 3$ and $|f(2)| = 3$ are equal — either works. We will use $x_0 = 2$ since $f(2) = 3$ is positive and close to zero.

$$x_0 = 2$$

**Step 3: Write the Newton's Method formula for this function.**

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 3}{3x_n^2 - 1}$$

**Step 4: Iteration 1 from $x_0 = 2$.**

$$f(2) = 3 \qquad f'(2) = 3(2)^2 - 1 = 12 - 1 = 11$$

$$x_1 = 2 - \frac{3}{11} \approx 2 - 0.27273 \approx 1.72727$$

**Step 5: Iteration 2 from $x_1 \approx 1.72727$.**

$$f(1.72727) \approx (1.72727)^3 - (1.72727) - 3 \approx 5.15531 - 1.72727 - 3 \approx 0.42804$$

$$f'(1.72727) = 3(1.72727)^2 - 1 \approx 3(2.98347) - 1 \approx 8.95041 - 1 = 7.95041$$

$$x_2 \approx 1.72727 - \frac{0.42804}{7.95041} \approx 1.72727 - 0.05385 \approx 1.67342$$

**Step 6: Iteration 3 from $x_2 \approx 1.67342$.**

$$f(1.67342) \approx (1.67342)^3 - (1.67342) - 3 \approx 4.68651 - 1.67342 - 3 \approx 0.01309$$

$$f'(1.67342) = 3(1.67342)^2 - 1 \approx 3(2.80033) - 1 \approx 8.40099 - 1 = 7.40099$$

$$x_3 \approx 1.67342 - \frac{0.01309}{7.40099} \approx 1.67342 - 0.001769 \approx 1.67165$$

$$\boxed{x \approx 1.6717}$$

**Example 4 (Choose your own $x_0$):** Use Newton's Method to approximate a root of $f(x) = x^2 - 3x - 1$. No starting guess is given. Choose one, then perform two iterations.

**Step 1: Find the derivative.**

$$f(x) = x^2 - 3x - 1 \qquad f'(x) = 2x - 3$$

**Step 2: Choose a starting guess by testing simple integers.**

$$f(3) = (3)^2 - 3(3) - 1 = 9 - 9 - 1 = -1 \quad \text{(negative)}$$

$$f(4) = (4)^2 - 3(4) - 1 = 16 - 12 - 1 = 3 \quad \text{(positive)}$$

There is a sign change between $x = 3$ and $x = 4$. We compare $|f(3)| = 1$ and $|f(4)| = 3$. Since $|f(3)|$ is smaller, the root is closer to $x = 3$.

$$x_0 = 3$$

**Step 3: Write the Newton's Method formula.**

$$x_{n+1} = x_n - \frac{x_n^2 - 3x_n - 1}{2x_n - 3}$$

**Step 4: Iteration 1 from $x_0 = 3$.**

$$f(3) = -1 \qquad f'(3) = 2(3) - 3 = 3$$

$$x_1 = 3 - \frac{-1}{3} = 3 + 0.3333 \approx 3.3333$$

**Step 5: Iteration 2 from $x_1 \approx 3.3333$.**

$$f(3.3333) \approx (3.3333)^2 - 3(3.3333) - 1 \approx 11.1109 - 9.9999 - 1 \approx 0.1110$$

$$f'(3.3333) = 2(3.3333) - 3 \approx 6.6666 - 3 = 3.6666$$

$$x_2 \approx 3.3333 - \frac{0.1110}{3.6666} \approx 3.3333 - 0.03028 \approx 3.30302$$

$$\boxed{x \approx 3.303}$$

# Common Mistakes

- **Forgetting to compute $f'(x)$ before starting:** You need both $f(x_n)$ and $f'(x_n)$ at every step. Find $f'(x)$ symbolically first; do not try to compute the derivative fresh each time you iterate.
- **Plugging $x_n$ into $f'$ incorrectly:** After finding $f'(x)$, make sure you substitute the current $x_n$ value, not the previous one. Each iteration uses the newest value.
- **Dividing in the wrong order:** The formula is $x_n - \dfrac{f(x_n)}{f'(x_n)}$, not $\dfrac{f'(x_n)}{f(x_n)}$. The function value goes on top.
- **Forgetting to subtract:** The formula subtracts the fraction. Watch your sign carefully, especially when $f(x_n)$ is negative, because subtracting a negative means adding.
- **Using the wrong starting guess:** If $x_0$ is far from the root, the method may converge slowly or fail entirely. Sketch the function or test a few values to find a starting point reasonably close to where $f(x)$ crosses zero.
- **Stopping too early:** One iteration is usually not enough for good accuracy. Perform at least two or three iterations unless the question specifies otherwise.

# Worksheet

1. Use Newton's Method with $x_0 = 2$ to approximate a root of $f(x) = x^2 - 7$. Perform **three** iterations.
   (Hint: you are approximating $\sqrt{7}$.)

2. Use Newton's Method with $x_0 = 1$ to approximate a root of $f(x) = x^2 - x - 2$. Perform **two** iterations.

3. Use Newton's Method with $x_0 = 1$ to approximate a root of $f(x) = x^3 - 2$. Perform **three** iterations.
(Hint: you are approximating $\sqrt[3]{2}$.)

4. Use Newton's Method with $x_0 = 0$ to approximate a root of $f(x) = e^x - 3$. Perform **two** iterations.

5. Use Newton's Method with $x_0 = 1$ to approximate a root of $f(x) = \ln(x) - 1$. Perform **two** iterations.
(Hint: you are approximating $e$.)

6. Use Newton's Method with $x_0 = 2$ to approximate a root of $f(x) = x^3 - 6$. Perform **three** iterations.

7. Use Newton's Method with $x_0 = 0.5$ to approximate a solution to $\sin(x) = x^2$. Perform **two** iterations.
(Hint: rewrite as $f(x) = \sin(x) - x^2 = 0$.)

8. (Challenge) A student uses Newton's Method on $f(x) = x^2 + 1$ starting at $x_0 = 1$. Perform one iteration and explain why this function has no real roots and why Newton's Method will still produce a value (but a meaningless one).

9. (Challenge) Use Newton's Method with $x_0 = 2$ to approximate a root of $f(x) = x^4 - 15$. Perform **three** iterations.

10. (Challenge) Explain in your own words the geometric meaning of each step in Newton's Method. What are we drawing? Where does the next guess come from?

# Answer Key

**1.** Newton's Method on $f(x) = x^2 - 7$, $x_0 = 2$, three iterations.

Step 1: Find the derivative.

$$f'(x) = 2x$$

Step 2: Iteration 1 from $x_0 = 2$.

$$f(2) = 4 - 7 = -3 \qquad f'(2) = 4$$

$$x_1 = 2 - \frac{-3}{4} = 2 + 0.75 = 2.75$$

Step 3: Iteration 2 from $x_1 = 2.75$.

$$f(2.75) = (2.75)^2 - 7 = 7.5625 - 7 = 0.5625 \qquad f'(2.75) = 5.5$$

$$x_2 = 2.75 - \frac{0.5625}{5.5} \approx 2.75 - 0.10227 \approx 2.64773$$

Step 4: Iteration 3 from $x_2 \approx 2.64773$.

$$f(2.64773) \approx (2.64773)^2 - 7 \approx 7.01047 - 7 = 0.01047$$

$$f'(2.64773) \approx 2(2.64773) = 5.29546$$

$$x_3 \approx 2.64773 - \frac{0.01047}{5.29546} \approx 2.64773 - 0.001977 \approx 2.645751$$

$$\boxed{\sqrt{7} \approx 2.6458}$$

**2.** Newton's Method on $f(x) = x^2 - x - 2$, $x_0 = 1$, two iterations.

Step 1: Find the derivative.

$$f'(x) = 2x - 1$$

Step 2: Iteration 1 from $x_0 = 1$.

$$f(1) = 1 - 1 - 2 = -2 \qquad f'(1) = 2(1) - 1 = 1$$

$$x_1 = 1 - \frac{-2}{1} = 1 + 2 = 3$$

Step 3: Iteration 2 from $x_1 = 3$.

$$f(3) = 9 - 3 - 2 = 4 \qquad f'(3) = 2(3) - 1 = 5$$

$$x_2 = 3 - \frac{4}{5} = 3 - 0.8 = 2.2$$

$$\boxed{x \approx 2.2}$$

*(The exact root is $x = 2$. With more iterations, Newton's Method converges to it.)*

**3.** Newton's Method on $f(x) = x^3 - 2$, $x_0 = 1$, three iterations.

Step 1: Find the derivative.

$$f'(x) = 3x^2$$

Step 2: Iteration 1 from $x_0 = 1$.

$$f(1) = 1 - 2 = -1 \qquad f'(1) = 3(1)^2 = 3$$

$$x_1 = 1 - \frac{-1}{3} = 1 + 0.3333 \approx 1.3333$$

Step 3: Iteration 2 from $x_1 \approx 1.3333$.

$$f(1.3333) \approx (1.3333)^3 - 2 \approx 2.3704 - 2 = 0.3704$$

$$f'(1.3333) = 3(1.3333)^2 \approx 3(1.7778) = 5.3334$$

$$x_2 \approx 1.3333 - \frac{0.3704}{5.3334} \approx 1.3333 - 0.06943 \approx 1.26387$$

Step 4: Iteration 3 from $x_2 \approx 1.26387$.

$$f(1.26387) \approx (1.26387)^3 - 2 \approx 2.01939 - 2 = 0.01939$$

$$f'(1.26387) = 3(1.26387)^2 \approx 3(1.59737) = 4.79211$$

$$x_3 \approx 1.26387 - \frac{0.01939}{4.79211} \approx 1.26387 - 0.004046 \approx 1.259824$$

$$\boxed{\sqrt[3]{2} \approx 1.2598}$$

**4.** Newton's Method on $f(x) = e^x - 3$, $x_0 = 0$, two iterations.

Step 1: Find the derivative. (Recall: the derivative of $e^x$ is $e^x$.)

$$f'(x) = e^x$$

Step 2: Iteration 1 from $x_0 = 0$.

$$f(0) = e^0 - 3 = 1 - 3 = -2 \qquad f'(0) = e^0 = 1$$

$$x_1 = 0 - \frac{-2}{1} = 0 + 2 = 2$$

Step 3: Iteration 2 from $x_1 = 2$.

$$f(2) = e^2 - 3 \approx 7.389 - 3 = 4.389 \qquad f'(2) = e^2 \approx 7.389$$

$$x_2 = 2 - \frac{4.389}{7.389} \approx 2 - 0.5940 \approx 1.406$$

$$\boxed{x \approx 1.406}$$

*(The exact answer is* $\ln 3 \approx 1.0986$*. The first guess of* $x_0 = 0$ *was far from the root, so more iterations would be needed.)*

**5.** Newton's Method on $f(x) = \ln(x) - 1$, $x_0 = 1$, two iterations.

Step 1: Find the derivative.

$$f'(x) = \frac{1}{x}$$

Step 2: Iteration 1 from $x_0 = 1$.

$$f(1) = \ln(1) - 1 = 0 - 1 = -1 \qquad f'(1) = \frac{1}{1} = 1$$

$$x_1 = 1 - \frac{-1}{1} = 1 + 1 = 2$$

Step 3: Iteration 2 from $x_1 = 2$.

$$f(2) = \ln(2) - 1 \approx 0.6931 - 1 = -0.3069 \qquad f'(2) = \frac{1}{2} = 0.5$$

$$x_2 = 2 - \frac{-0.3069}{0.5} = 2 + 0.6138 = 2.6138$$

$$\boxed{e \approx 2.6138}$$

*(The actual value of* $e \approx 2.71828$*. With more iterations this converges to* $e$*.)*

**6.** Newton's Method on $f(x) = x^3 - 6$, $x_0 = 2$, three iterations.

Step 1: Find the derivative.

$$f'(x) = 3x^2$$

Step 2: Iteration 1 from $x_0 = 2$.

$$f(2) = 8 - 6 = 2 \qquad f'(2) = 3(4) = 12$$

$$x_1 = 2 - \frac{2}{12} = 2 - 0.1667 \approx 1.8333$$

Step 3: Iteration 2 from $x_1 \approx 1.8333$.

$$f(1.8333) \approx (1.8333)^3 - 6 \approx 6.1614 - 6 = 0.1614$$

$$f'(1.8333) = 3(1.8333)^2 \approx 3(3.3610) = 10.0830$$

$$x_2 \approx 1.8333 - \frac{0.1614}{10.0830} \approx 1.8333 - 0.016 \approx 1.8173$$

Step 4: Iteration 3 from $x_2 \approx 1.8173$.

$$f(1.8173) \approx (1.8173)^3 - 6 \approx 6.00262 - 6 = 0.00262$$

$$f'(1.8173) = 3(1.8173)^2 \approx 3(3.3026) = 9.9078$$

$$x_3 \approx 1.8173 - \frac{0.00262}{9.9078} \approx 1.8173 - 0.000264 \approx 1.81703$$

$$\boxed{\sqrt[3]{6} \approx 1.8171}$$

**7.** Newton's Method on $f(x) = \sin(x) - x^2$, $x_0 = 0.5$, two iterations.

Step 1: Find the derivative.

$$f'(x) = \cos(x) - 2x$$

Step 2: Iteration 1 from $x_0 = 0.5$.

$$f(0.5) = \sin(0.5) - (0.5)^2 \approx 0.4794 - 0.25 = 0.2294$$

$$f'(0.5) = \cos(0.5) - 2(0.5) \approx 0.8776 - 1 = -0.1224$$

$$x_1 = 0.5 - \frac{0.2294}{-0.1224} \approx 0.5 + 1.8742 \approx 2.3742$$

*Note: The large jump tells us $x_0 = 0.5$ is near a point where the tangent is nearly flat. The other real solution to $\sin(x) = x^2$ is $x = 0$ (trivial). For the non-zero root, a better starting guess is $x_0 = 0.8$.*

Step 3: Redo with $x_0 = 0.8$.

$$f(0.8) = \sin(0.8) - (0.8)^2 \approx 0.7174 - 0.64 = 0.0774$$

$$f'(0.8) = \cos(0.8) - 2(0.8) \approx 0.6967 - 1.6 = -0.9033$$

$$x_1 = 0.8 - \frac{0.0774}{-0.9033} \approx 0.8 + 0.08568 \approx 0.88568$$

$$f(0.88568) \approx \sin(0.88568) - (0.88568)^2 \approx 0.77466 - 0.78443 = -0.00977$$

$$f'(0.88568) \approx \cos(0.88568) - 2(0.88568) \approx 0.63245 - 1.77136 = -1.13891$$

$$x_2 \approx 0.88568 - \frac{-0.00977}{-1.13891} \approx 0.88568 - 0.00858 \approx 0.87710$$

$$\boxed{x \approx 0.877}$$

**8.** (Challenge) Newton's Method on $f(x) = x^2 + 1$, $x_0 = 1$, one iteration.

Step 1: Find the derivative.

$$f'(x) = 2x$$

Step 2: Perform one iteration.

$$f(1) = (1)^2 + 1 = 2 \qquad f'(1) = 2(1) = 2$$

$$x_1 = 1 - \frac{2}{2} = 1 - 1 = 0$$

Step 3: Explain the situation.

The function $f(x) = x^2 + 1$ is always positive (the minimum value is 1, at $x = 0$). It *never* crosses the $x$-axis, so it has no real roots. Newton's Method still produces values like $x_1 = 0$, but these values have no meaning as roots. The tangent lines never actually reach a root because no root exists.

**Lesson:** Always verify that the function actually has a real root before applying Newton's Method. A quick check is to find values of $x$ where $f(x)$ is positive and negative — if both exist, a root exists between them by the Intermediate Value Theorem.

**9.** (Challenge) Newton's Method on $f(x) = x^4 - 15$, $x_0 = 2$, three iterations.

Step 1: Find the derivative.

$$f'(x) = 4x^3$$

Step 2: Iteration 1 from $x_0 = 2$.

$$f(2) = 16 - 15 = 1 \qquad f'(2) = 4(8) = 32$$

$$x_1 = 2 - \frac{1}{32} = 2 - 0.03125 = 1.96875$$

Step 3: Iteration 2 from $x_1 = 1.96875$.

$$f(1.96875) = (1.96875)^4 - 15 \approx 15.0072 - 15 = 0.0072$$

$$f'(1.96875) = 4(1.96875)^3 \approx 4(7.6294) = 30.5176$$

$$x_2 \approx 1.96875 - \frac{0.0072}{30.5176} \approx 1.96875 - 0.000236 \approx 1.96851$$

Step 4: Iteration 3 from $x_2 \approx 1.96851$.

$$f(1.96851) \approx (1.96851)^4 - 15 \approx 14.99998 - 15 \approx -0.00002$$

$$f'(1.96851) = 4(1.96851)^3 \approx 30.5068$$

$$x_3 \approx 1.96851 - \frac{-0.00002}{30.5068} \approx 1.96851 + 0.000001 \approx 1.96851$$

$$\boxed{15^{1/4} \approx 1.9685}$$

**10.** (Challenge) Geometric meaning of Newton's Method.

Step 1: What are we drawing at each step?

At the current guess $x_n$, we locate the point on the curve $(x_n, f(x_n))$. We then draw the **tangent line** to the curve at that point. This tangent line has slope $f'(x_n)$.

Step 2: Where does the next guess come from?

The tangent line is straight, so it is easy to find where it crosses the $x$-axis (where $y = 0$). That crossing point becomes our next guess $x_{n+1}$.

Step 3: Why does this improve the guess?

Near $x_n$, the tangent line is a good approximation of the curve (just like linearization). So the point where the *line* hits zero is close to the point where the *curve* hits zero. With each iteration, we start closer to the root, the tangent line is even more accurate, and our next guess is better still.

**Summary:** Each step of Newton's Method means: find the tangent line at your current guess, then slide along that line to where it hits the $x$-axis. That intersection is your improved guess.

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