Lesson 4: One-Sided Limits and Limits at Infinity

ApexMath

Notes on One-Sided Limits and Limits at Infinity

- **One-sided limits:**

$$\lim_{x \to a^+} f(x) \text{ (from the right)}, \quad \lim_{x \to a^-} f(x) \text{ (from the left)}$$

- A two-sided limit exists if and only if the left- and right-hand limits are equal.
- **Limits at infinity:**

$$\lim_{x \to \infty} f(x), \quad \lim_{x \to -\infty} f(x)$$

  - Horizontal asymptotes occur where the limit approaches a finite value. - For rational functions, compare degrees of numerator and denominator:
    - Degree numerator $<$ degree denominator $\to$ limit $= 0$
    - Degree numerator $=$ degree denominator $\to$ limit $=$ ratio of leading coefficients
    - Degree numerator $>$ degree denominator $\to$ limit $= \infty$ or $-\infty$

Deep Walkthrough Examples

Example A. Evaluate $\lim_{x \to 2^+} \frac{1}{x-2}$

Step 1. Consider values slightly greater than 2: $x = 2.01, 2.001, ...$

Step 2. Compute $1/(x-2) \to$ positive numbers, increasing without bound.

Step 3. Conclude:

$$\lim_{x \to 2^+} \frac{1}{x-2} = \infty$$

$$\boxed{\text{Infinity}}$$

—

Example B. Evaluate $\lim_{x \to 2^-} \frac{1}{x-2}$

Step 1. Consider values slightly less than 2: $x = 1.99, 1.999, ...$

Step 2. Compute $1/(x-2) \to$ negative numbers, decreasing without bound.

Step 3. Conclude:

$$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$$

$$\boxed{\text{Negative infinity}}$$

—

Example C. Evaluate $\lim_{x\to\infty} \frac{3x^2+5}{2x^2-x+1}$

Step 1. Compare degrees: numerator degree = 2, denominator degree = 2

Step 2. Ratio of leading coefficients: 3/2

Step 3. Conclude:

$$\lim_{x\to\infty} \frac{3x^2+5}{2x^2-x+1} = \frac{3}{2}$$

$$\boxed{3/2}$$

—

Example D. Evaluate $\lim_{x\to-\infty} \frac{x^3-2x}{5x^4+1}$

Step 1. Compare degrees: numerator = 3, denominator = 4 → numerator degree ¡ denominator degree

Step 2. Conclude:

$$\lim_{x\to-\infty} \frac{x^3-2x}{5x^4+1} = 0$$

$$\boxed{0}$$

—

Common Mistakes

- Forgetting that one-sided limits may differ from each other

- Confusing positive infinity with negative infinity when approaching from left/right

- Not comparing degrees correctly in rational functions

- Forgetting horizontal asymptote rules

- Mixing up infinity as a number versus a concept

Worksheet Problems

1. Evaluate: $\lim_{x\to1^+} \frac{2}{x-1}$
2. Evaluate: $\lim_{x\to1^-} \frac{2}{x-1}$
3. Find the end behaviors of the function $f(x) = \frac{5x^3-2}{2x^3+7}$.
4. Evaluate: $\lim_{x\to-\infty} \frac{4x^2-x}{7x^3+2}$
5. Evaluate: $\lim_{x\to\infty} \frac{6x^4-x^2+1}{3x^4+5x^2-2}$
6. Evaluate: $\lim_{x\to-\infty} \frac{x^5-2x^2}{x^5+x^3-1}$

7. Evaluate: $\lim_{x\to\infty} \frac{7x^2+3}{x^3+5}$
8. Evaluate: $\lim_{x\to 0^+} \frac{\ln x}{x}$ (challenging)

Answer Key

1. $\lim_{x\to 1^+} \frac{2}{x-1}$

Step 1: Approach 1 from the right ($x > 1$).

Step 2: Denominator is positive → fraction increases → $+\infty$.

$$\boxed{\text{Infinity}}$$

—

2. $\lim_{x\to 1^-} \frac{2}{x-1}$

Step 1: Approach 1 from the left ($x < 1$).

Step 2: Denominator is negative → fraction decreases → $-\infty$.

$$\boxed{\text{Negative infinity}}$$

—

3. End behaviors of $f(x) = \frac{5x^3-2}{2x^3+7}$

Step 1: As $x \to \infty$, divide numerator and denominator by x^3:

$$f(x) = \frac{5x^3-2}{2x^3+7} = \frac{\frac{5x^3}{x^3} - \frac{2}{x^3}}{\frac{2x^3}{x^3} + \frac{7}{x^3}} = \frac{5 - \frac{2}{x^3}}{2 + \frac{7}{x^3}}$$

Step 2: As $x \to \infty$, $\frac{2}{x^3} \to 0$ and $\frac{7}{x^3} \to 0$.

Step 3: Simplify the limit:

$$\lim_{x\to\infty} f(x) = \frac{5-0}{2+0} = \frac{5}{2}$$

Step 4: As $x \to -\infty$, same process:

$$\lim_{x\to-\infty} f(x) = \frac{5-0}{2+0} = \frac{5}{2}$$

$$\boxed{\text{Horizontal asymptote: y = 5/2}}$$

—

4. $\lim_{x\to-\infty} \frac{4x^2-x}{7x^3+2}$

Step 1: Degree numerator ¡ degree denominator → limit = 0.

$$\boxed{0}$$

—

5. $\lim_{x\to\infty} \frac{6x^4-x^2+1}{3x^4+5x^2-2}$

Step 1: Degree numerator = degree denominator → ratio of leading coefficients = 6/3.

$\boxed{2}$

—

6. $\lim_{x\to-\infty} \frac{x^5-2x^2}{x^5+x^3-1}$

Step 1: Degree numerator = degree denominator → ratio of leading coefficients = 1/1.

$\boxed{1}$

—

7. $\lim_{x\to\infty} \frac{7x^2+3}{x^3+5}$

Step 1: Degree numerator ¡ degree denominator → limit = 0.

$\boxed{0}$

—

8. $\lim_{x\to0^+} \frac{\ln x}{x}$

Step 1: As $x \to 0^+$, $\ln x \to -\infty$, $x \to 0^+$.

Step 2: Fraction = $-\infty/0^+ \to -\infty$.

$\boxed{\text{Negative infinity}}$

Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

End of Lesson • ApexMath
