Properties of Definite Integrals
ApexMath

Notes

Why Do These Properties Matter?

So far we have evaluated definite integrals by finding antiderivatives and applying FTC Part 2. But sometimes we are given information about integrals — specific values — and need to find a related integral without computing anything from scratch. The properties of definite integrals let us manipulate, combine, and relate integral expressions algebraically, just like the rules of arithmetic let us rearrange numbers.

Property 1: Zero-Width Interval

If the upper and lower limits are the same, the interval has no width and the area is zero:

$$\int_a^a f(x)\,dx = 0$$

This makes sense geometrically: a region with no width has no area.

Property 2: Reversing the Limits

Swapping the upper and lower limits negates the integral:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Think of it as integrating "backwards" across the interval, which flips the sign of the area.

Property 3: Constant Multiple Rule

A constant factor can be pulled out of a definite integral:

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

Property 4: Sum and Difference Rule

A definite integral of a sum or difference can be split into separate integrals over the same interval:

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Property 5: Additivity (Splitting the Interval)

If c is any point, the integral over $[a, b]$ can be split into two pieces at c:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

This works even if c is outside $[a, b]$. The property is extremely useful when a function changes its formula at some point inside the interval, or when you need to combine two known integrals into one.

Property 6: Comparison Properties

These properties connect the sign and size of an integral to the behavior of f:

If $f(x) \geq 0$ for all x in $[a, b]$, then:

$$\int_a^b f(x)\, dx \geq 0$$

If $f(x) \geq g(x)$ for all x in $[a, b]$, then:

$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$$

Property 7: Bounding the Integral

If f has a minimum value m and a maximum value M on $[a, b]$, meaning $m \leq f(x) \leq M$, then:

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$$

In plain words: the integral is at least as large as the minimum value times the width of the interval, and at most as large as the maximum value times the width. This lets you estimate an integral without computing it exactly.

Summary Table

Property	**Rule**
Zero-width interval	$\int_a^a f(x)\, dx = 0$
Reversing limits	$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
Constant multiple	$\int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx$
Sum/difference	$\int_a^b [f \pm g]\, dx = \int_a^b f\, dx \pm \int_a^b g\, dx$
Additivity	$\int_a^b f\, dx = \int_a^c f\, dx + \int_c^b f\, dx$
Comparison	$f(x) \geq g(x) \Rightarrow \int_a^b f\, dx \geq \int_a^b g\, dx$
Bounding	$m(b-a) \leq \int_a^b f\, dx \leq M(b-a)$

Pro Tip

- Properties 3 and 4 together are called **linearity** — the same linearity you used for indefinite integrals. The rules carry over directly to definite integrals, with the same limits on every piece.
- When using the additivity property to split an integral, make sure the limits chain correctly: the right limit of the first piece must equal the left limit of the second piece.
- The reversing-limits property is especially useful when a problem gives you $\displaystyle\int_b^a$ but you know $\displaystyle\int_a^b$. Just flip the sign.
- For the bounding property, find m and M by identifying the minimum and maximum values of f on $[a, b]$. For a continuous function on a closed interval, you can find these using critical points or just by evaluating at the endpoints if the function is monotone.
- Problems that say "given that $\int_a^b f(x)\,dx = k$, find…" are almost always testing Properties 2–5. Look for constants being pulled out, limits being reversed, or intervals being split or combined.

Examples

Example 1 (Additivity): Given that $\displaystyle\int_1^5 f(x)\,dx = 10$ and $\displaystyle\int_1^3 f(x)\,dx = 4$, find $\displaystyle\int_3^5 f(x)\,dx$.

Step 1: Write the additivity property.
By Property 5:

$$\int_1^5 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

Step 2: Substitute the known values.

$$10 = 4 + \int_3^5 f(x)\,dx$$

Step 3: Solve for the unknown integral.

$$\int_3^5 f(x)\,dx = 10 - 4 = 6$$

$$\boxed{\int_3^5 f(x)\,dx = 6}$$

Example 2 (Constant Multiple and Sum): Given $\int_0^4 f(x)\,dx = 7$ and $\int_0^4 g(x)\,dx = 3$, find $\int_0^4 [5f(x) - 2g(x)]\,dx$.

Step 1: Apply the sum/difference rule to split the integral.

$$\int_0^4 [5f(x) - 2g(x)]\,dx = \int_0^4 5f(x)\,dx - \int_0^4 2g(x)\,dx$$

Step 2: Apply the constant multiple rule to pull out the constants.

$$= 5\int_0^4 f(x)\,dx - 2\int_0^4 g(x)\,dx$$

Step 3: Substitute the known values.

$$= 5(7) - 2(3) = 35 - 6 = 29$$

$$\boxed{\int_0^4 [5f(x) - 2g(x)]\,dx = 29}$$

Example 3 (Reversing Limits): Given $\int_2^6 f(x)\,dx = -5$, find $\int_6^2 [3f(x)]\,dx$.

Step 1: Pull out the constant.

$$\int_6^2 3f(x)\,dx = 3\int_6^2 f(x)\,dx$$

Step 2: Reverse the limits and flip the sign.

$$3\int_6^2 f(x)\,dx = 3\left(-\int_2^6 f(x)\,dx\right) = -3\int_2^6 f(x)\,dx$$

Step 3: Substitute the known value.

$$= -3(-5) = 15$$

$$\boxed{\int_6^2 3f(x)\,dx = 15}$$

Example 4 (Bounding): Use the bounding property to estimate $\int_0^3 e^x\,dx$ without computing it exactly.

Step 1: Find the minimum and maximum of $f(x) = e^x$ on $[0, 3]$.
Since e^x is an increasing function:

$$m = e^0 = 1 \qquad M = e^3 \approx 20.09$$

Step 2: Apply the bounding property. The interval width is $b - a = 3 - 0 = 3$.

$$m(b-a) \leq \int_0^3 e^x \, dx \leq M(b-a)$$

$$1 \cdot 3 \leq \int_0^3 e^x \, dx \leq e^3 \cdot 3$$

$$\boxed{3 \leq \int_0^3 e^x \, dx \leq 3e^3}$$

(The exact value is $e^3 - 1 \approx 19.09$, which is indeed between 3 and $3e^3 \approx 60.3$.)

Example 5 (Multiple Properties Combined): Given:

$$\int_0^5 f(x) \, dx = 12, \qquad \int_5^8 f(x) \, dx = -3, \qquad \int_0^8 g(x) \, dx = 6$$

Find $\int_0^8 [2f(x) + g(x)] \, dx$.

Step 1: Split using the sum rule.

$$\int_0^8 [2f(x) + g(x)] \, dx = 2 \int_0^8 f(x) \, dx + \int_0^8 g(x) \, dx$$

Step 2: We do not directly know $\int_0^8 f(x) \, dx$, so use additivity to find it.

$$\int_0^8 f(x) \, dx = \int_0^5 f(x) \, dx + \int_5^8 f(x) \, dx = 12 + (-3) = 9$$

Step 3: Substitute all known values.

$$= 2(9) + 6 = 18 + 6 = 24$$

$$\boxed{\int_0^8 [2f(x) + g(x)] \, dx = 24}$$

Common Mistakes

- **Forgetting to flip the sign when reversing limits:** $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$. A very common error is to copy the value without changing the sign.
- **Applying the sum rule across different intervals:** The sum rule requires both integrals to share the *same* limits a and b. You cannot split $\int_0^3 f(x)\,dx + \int_1^5 g(x)\,dx$ using Property 4 — the intervals are different.
- **Misapplying additivity:** When splitting $\int_a^c = \int_a^b + \int_b^c$, the split point b must appear as the right limit of the first piece and the left limit of the second piece. A common error is writing $\int_a^b + \int_a^c$ or getting the endpoints out of order.
- **Pulling a non-constant outside the integral:** Only *constants* (numbers that do not depend on x) can be pulled out. You cannot move $f(x)$ outside or split a product $f(x) \cdot g(x)$ using these properties.
- **Confusing the comparison property direction:** If $f(x) \geq g(x)$, then $\int f \geq \int g$. The inequality goes the same way as the function comparison. It does *not* flip.

Worksheet

1. Given $\int_1^6 f(x)\,dx = 15$ and $\int_4^6 f(x)\,dx = 5$, find $\int_1^4 f(x)\,dx$.

2. Given $\int_0^3 f(x)\,dx = 8$ and $\int_0^3 g(x)\,dx = -2$, find $\int_0^3 [4f(x) + 3g(x)]\,dx$.

3. Given $\displaystyle\int_2^9 f(x)\,dx = 11$, find $\displaystyle\int_9^2 f(x)\,dx$.

4. Given $\displaystyle\int_2^9 f(x)\,dx = 11$, find $\displaystyle\int_9^2 [-6f(x)]\,dx$.

5. Given $\displaystyle\int_0^2 f(x)\,dx = 5$ and $\displaystyle\int_2^7 f(x)\,dx = 9$, find $\displaystyle\int_0^7 f(x)\,dx$.

6. Given $\displaystyle\int_0^6 f(x)\,dx = 20$ and $\displaystyle\int_0^2 f(x)\,dx = 7$, find $\displaystyle\int_2^6 f(x)\,dx$.

7. Use the bounding property to estimate $\displaystyle\int_1^4 \sqrt{x}\,dx$.
   (Find the min and max of $\sqrt{x}$ on $[1,4]$, then apply $m(b-a) \le \int \le M(b-a)$.)

8. Use the bounding property to estimate $\int_0^{\pi/2} \sin(x)\,dx$.

9. (Challenge) Given:

$$\int_0^3 f(x)\,dx = 6, \qquad \int_3^7 f(x)\,dx = -4, \qquad \int_0^7 g(x)\,dx = 10$$

Find $\int_0^7 [3f(x) - 2g(x)]\,dx$.

10. (Challenge) Suppose $2 \leq f(x) \leq 6$ for all x in $[1,5]$.

    (a) Use the bounding property to give upper and lower bounds for $\int_1^5 f(x)\,dx$.

    (b) Is it possible that $\int_1^5 f(x)\,dx = 15$? Explain.

    (c) Is it possible that $\int_1^5 f(x)\,dx = 9$? Explain.

Answer Key

1. Find $\int_1^4 f(x)\,dx$ given $\int_1^6 f(x)\,dx = 15$ and $\int_4^6 f(x)\,dx = 5$.
By additivity:

$$\int_1^6 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^6 f(x)\,dx$$

$$15 = \int_1^4 f(x)\,dx + 5$$

$$\int_1^4 f(x)\,dx = 15 - 5 = 10$$

$$\boxed{\int_1^4 f(x)\,dx = 10}$$

2. Find $\int_0^3 [4f(x) + 3g(x)]\,dx$ given $\int_0^3 f(x)\,dx = 8$ and $\int_0^3 g(x)\,dx = -2$.
Step 1: Apply sum rule and constant multiple rule.

$$\int_0^3 [4f(x) + 3g(x)]\,dx = 4\int_0^3 f(x)\,dx + 3\int_0^3 g(x)\,dx$$

Step 2: Substitute.

$$= 4(8) + 3(-2) = 32 - 6 = 26$$

$$\boxed{\int_0^3 [4f(x) + 3g(x)]\,dx = 26}$$

3. Find $\int_9^2 f(x)\,dx$ given $\int_2^9 f(x)\,dx = 11$.
By the reversing limits property:

$$\int_9^2 f(x)\,dx = -\int_2^9 f(x)\,dx = -11$$

$$\boxed{\int_9^2 f(x)\,dx = -11}$$

4. Find $\int_9^2 [-6f(x)]\,dx$ given $\int_2^9 f(x)\,dx = 11$.
Step 1: Pull out the constant.

$$\int_9^2 -6f(x)\,dx = -6\int_9^2 f(x)\,dx$$

Step 2: Reverse the limits.

$$= -6\left(-\int_2^9 f(x)\,dx\right) = 6\int_2^9 f(x)\,dx$$

Step 3: Substitute.

$$= 6(11) = 66$$

$$\boxed{\int_9^2 [-6f(x)]\,dx = 66}$$

5. Find $\int_0^7 f(x)\,dx$ given $\int_0^2 f(x)\,dx = 5$ and $\int_2^7 f(x)\,dx = 9$.
By additivity:

$$\int_0^7 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^7 f(x)\,dx = 5 + 9 = 14$$

$$\boxed{\int_0^7 f(x)\,dx = 14}$$

6. Find $\int_2^6 f(x)\,dx$ given $\int_0^6 f(x)\,dx = 20$ and $\int_0^2 f(x)\,dx = 7$.
By additivity:

$$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx$$

$$20 = 7 + \int_2^6 f(x)\,dx$$

$$\int_2^6 f(x)\,dx = 20 - 7 = 13$$

$$\boxed{\int_2^6 f(x)\,dx = 13}$$

7. Estimate $\int_1^4 \sqrt{x}\,dx$ using the bounding property.
Step 1: Find m and M on $[1, 4]$.
Since $\sqrt{x}$ is increasing on $[1, 4]$:

$$m = \sqrt{1} = 1 \qquad M = \sqrt{4} = 2$$

Step 2: Interval width is $b - a = 4 - 1 = 3$.
Step 3: Apply the bounding property.

$$1 \cdot 3 \le \int_1^4 \sqrt{x}\,dx \le 2 \cdot 3$$

$$\boxed{3 \leq \int_1^4 \sqrt{x}\,dx \leq 6}$$

(The exact value is $\frac{2}{3}(4^{3/2} - 1^{3/2}) = \frac{2}{3}(8-1) = \frac{14}{3} \approx 4.67$*, which is indeed between 3 and 6.)*

8. Estimate $\displaystyle\int_0^{\pi/2} \sin(x)\,dx$ using the bounding property.
Step 1: Find m and M on $[0, \pi/2]$.
$\sin(x)$ is increasing on $[0, \pi/2]$, so:

$$m = \sin(0) = 0 \qquad M = \sin\left(\frac{\pi}{2}\right) = 1$$

Step 2: Interval width is $b - a = \dfrac{\pi}{2} - 0 = \dfrac{\pi}{2}$.
Step 3: Apply the bounding property.

$$0 \cdot \frac{\pi}{2} \leq \int_0^{\pi/2} \sin(x)\,dx \leq 1 \cdot \frac{\pi}{2}$$

$$\boxed{0 \leq \int_0^{\pi/2} \sin(x)\,dx \leq \frac{\pi}{2}}$$

(The exact value is 1*, which is between* 0 *and* $\frac{\pi}{2} \approx 1.571$*.)*

9. (Challenge) Find $\displaystyle\int_0^7 [3f(x) - 2g(x)]\,dx$.
Step 1: Apply linearity.

$$\int_0^7 [3f(x) - 2g(x)]\,dx = 3\int_0^7 f(x)\,dx - 2\int_0^7 g(x)\,dx$$

Step 2: Find $\displaystyle\int_0^7 f(x)\,dx$ using additivity.

$$\int_0^7 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx = 6 + (-4) = 2$$

Step 3: Substitute all known values.

$$= 3(2) - 2(10) = 6 - 20 = -14$$

$$\boxed{\int_0^7 [3f(x) - 2g(x)]\,dx = -14}$$

10. (Challenge) $2 \leq f(x) \leq 6$ on $[1, 5]$, so $m = 2$, $M = 6$, and $b - a = 4$.

(a) Apply the bounding property.

$$2(4) \leq \int_1^5 f(x)\,dx \leq 6(4)$$

$$\boxed{8 \leq \int_1^5 f(x)\,dx \leq 24}$$

(b) Is $\displaystyle\int_1^5 f(x)\,dx = 15$ possible?

Yes. The value 15 lies within the bounds $[8, 24]$, so it is entirely possible for the integral to equal 15. The bounding property only tells us the integral must fall between 8 and 24 — any value in that range is achievable by some function satisfying $2 \leq f(x) \leq 6$.

(c) Is $\displaystyle\int_1^5 f(x)\,dx = 9$ possible?

Yes. The value 9 also lies within $[8, 24]$, so it is possible as well. For example, a function that is very close to 2 across most of the interval could produce an integral near 8.

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