Lesson 12: Real-World Optimization Problems

ApexMath

Notes

- Optimization problems find **maximum or minimum** values (area, volume, cost, distance, etc.).
- Express the quantity to optimize as a **function of one variable**.
- Steps to solve:
  1. Identify the quantity to optimize.
  2. Define variables and relationships.
  3. Use substitution to write the function in *one variable*.
  4. Differentiate to find critical points.
  5. Use **second derivative test** or logic to determine max/min.
  6. Interpret the result in context with proper units.
- Draw a diagram if possible.
- Clearly label variables.
- Check that the answer is physically reasonable (e.g., no negative lengths).
- Double-check which quantity is being optimized.

Pro Tips

- Always draw a diagram and label variables clearly.
- Use substitution carefully to reduce to one variable.
- Check your critical points against endpoints if applicable.
- Make sure your units match the problem context.
- Verify that your solution makes sense (e.g., lengths must be positive).

Examples

Example 1: Fence Problem A farmer has 100 m of fencing and wants to build a rectangular pen against a wall (no fence along the wall). Find dimensions that maximize the area.

Step 1 – Define variables:

$$\text{Let } x = \text{length perpendicular to wall}, \ y = \text{length along wall}$$

Step 2 – Express constraint:

$$2x + y = 100 \implies y = 100 - 2x$$

Step 3 – Area function:

$$A = x \cdot y = x(100 - 2x) = 100x - 2x^2$$

Step 4 – Derivative:

$$A' = 100 - 4x$$

Step 5 – Critical point:

$$100 - 4x = 0 \implies x = 25$$

Step 6 – Second derivative:

$$A'' = -4 < 0 \implies \text{maximum}$$

Step 7 – Solve for y:

$$y = 100 - 2(25) = 50$$

Step 8 – Answer:

$$\boxed{x = 25 \text{ m}, \ y = 50 \text{ m; max area } 1250 \text{ m}^2}$$

Example 2: Box in a Circle Problem Find the maximum area of a rectangle inscribed in a circle of radius $r = 5$.

Step 1 – Let variables:

$$x = \text{half-width}, \ y = \text{half-height}, \ \text{so rectangle area } A = (2x)(2y) = 4xy$$

Step 2 – Circle constraint:

$$x^2 + y^2 = 25 \implies y = \sqrt{25 - x^2}$$

Step 3 – Area function in one variable:

$$A(x) = 4x\sqrt{25 - x^2}$$

Step 4 – Derivative (product rule):

$$A'(x) = 4\sqrt{25 - x^2} + 4x \cdot \frac{-x}{\sqrt{25 - x^2}} = \frac{100 - 8x^2}{\sqrt{25 - x^2}}$$

Step 5 – Solve for critical points:

$$100 - 8x^2 = 0 \implies x^2 = 12.5 \implies x = \sqrt{12.5} \approx 3.536$$

Step 6 – Solve for y:

$$y = \sqrt{25 - 12.5} = \sqrt{12.5} \approx 3.536$$

Step 7 – Second derivative test:

$$A''(x) < 0 \implies \text{maximum area}$$

Step 8 – Answer:

$$\boxed{\text{Square with sides } 2x = 2y \approx 7.07 \text{ m; max area } \approx 50 \text{ m}^2}$$

Example 3: Poster with margins A poster must have printed area 200 in^2 with 2-inch margins on all sides. Minimize total paper used.

Step 1 – Variables:

$$w = \text{width of printed area}, \; h = \text{height of printed area}, \; wh = 200$$

Step 2 – Solve for h:

$$h = \frac{200}{w}$$

Step 3 – Total dimensions including margins:

$$W = w + 4, \quad H = h + 4$$

Step 4 – Area function:

$$A = W \cdot H = (w + 4)\left(\frac{200}{w} + 4\right) = 4w + \frac{800}{w} + 216$$

Step 5 – Derivative:

$$A' = 4 - \frac{800}{w^2}$$

Step 6 – Critical point:

$$4 - \frac{800}{w^2} = 0 \implies w^2 = 200 \implies w \approx 14.142$$

Step 7 – Solve for h:

$$h = \frac{200}{14.142} \approx 14.142$$

Step 8 – Answer:

$$\boxed{W \approx 18.142 \text{ in}, \; H \approx 18.142 \text{ in; minimal paper used}}$$

Common Mistakes

- Forgetting to reduce the function to one variable.
- Ignoring the constraint equation.
- Not checking critical points against endpoints.
- Forgetting to interpret the result in context (units, physical meaning).
- Misusing the second derivative test or ignoring negative dimensions.

Worksheet

1. A farmer has 120 m of fencing and wants to build a rectangular pen against a wall. Find the dimensions that maximize area.
2. Find the dimensions of a rectangle with maximum area that can be inscribed in a semicircle of radius 6.
3. A box with a square base and open top must have a volume of 32 m^3. Find dimensions that minimize surface area.
4. Find the maximum area of a rectangle inscribed in a circle of radius 10.
5. A poster must have area 200 in^2 with 2-inch margins on all sides. Find dimensions that minimize paper used.

Answer Key

1. **Question 1 – Fence Problem**

   Step 1 – Variables:

   $$x = \text{perpendicular length}, \; y = \text{along wall}$$

   Step 2 – Constraint:

   $$2x + y = 120 \implies y = 120 - 2x$$

   Step 3 – Area:

   $$A = x \cdot y = x(120 - 2x) = 120x - 2x^2$$

   Step 4 – Derivative:

   $$A' = 120 - 4x$$

   Step 5 – Critical point:

   $$120 - 4x = 0 \implies x = 30$$

Step 6 – Second derivative:

$$A'' = -4 < 0 \implies \text{maximum}$$

Step 7 – Solve for y:

$$y = 120 - 2(30) = 60$$

Step 8 – Answer:

$$\boxed{x = 30 \text{ m}, y = 60 \text{ m; max area } 1800 \text{ m}^2}$$

Question 2 – Rectangle in semicircle

Step 1 – Variables:

$$x = \text{half-width}, \ y = \text{height}, A = 2x \cdot y = 2xy$$

Step 2 – Constraint:

$$x^2 + y^2 = 36 \implies y = \sqrt{36 - x^2}$$

Step 3 – Area function:

$$A = 2x\sqrt{36 - x^2}$$

Step 4 – Derivative:

$$A' = 2\sqrt{36 - x^2} + 2x \cdot \frac{-x}{\sqrt{36 - x^2}} = \frac{72 - 4x^2}{\sqrt{36 - x^2}}$$

Step 5 – Critical point:

$$72 - 4x^2 = 0 \implies x^2 = 18 \implies x = \sqrt{18} \approx 4.243$$

Step 6 – Solve for y:

$$y = \sqrt{36 - 18} = \sqrt{18} \approx 4.243$$

Step 7 – Second derivative: $A'' < 0 \implies$ maximum **Step 8 – Answer:**

$$\boxed{2x = 2y \approx 8.486 \text{ m; max area } \approx 36 \text{ m}^2}$$

Question 3 – Open box with square base

Step 1 – Variables:

$$x = \text{side of base}, h = \text{height}, V = x^2 h = 32 \implies h = \frac{32}{x^2}$$

Step 2 – Surface area:

$$S = x^2 + 4(x \cdot h) = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x}$$

Step 3 – Derivative:

$$S' = 2x - \frac{128}{x^2}$$

Step 4 – Critical point:

$$2x - \frac{128}{x^2} = 0 \implies 2x^3 = 128 \implies x = 4$$

Step 5 – Solve for h:

$$h = \frac{32}{4^2} = 2$$

Step 6 – Second derivative:

$$S'' = 2 + \frac{256}{x^3} = 2 + 4 = 6 > 0 \implies \text{minimum}$$

Step 7 – Answer:

$$\boxed{x = 4 \text{ m}, h = 2 \text{ m; min surface area}}$$

Question 4 – Rectangle in circle radius 10
Step 1 – Variables:

$$x = \text{half-width}, \ y = \text{half-height}, A = 4xy$$

Step 2 – Constraint:

$$x^2 + y^2 = 100 \implies y = \sqrt{100 - x^2}$$

Step 3 – Area function:

$$A = 4x\sqrt{100 - x^2}$$

Step 4 – Derivative:

$$A' = 4\sqrt{100 - x^2} + 4x \cdot \frac{-x}{\sqrt{100 - x^2}} = \frac{400 - 8x^2}{\sqrt{100 - x^2}}$$

Step 5 – Critical point:

$$400 - 8x^2 = 0 \implies x^2 = 50 \implies x \approx 7.071$$

Step 6 – Solve for y:

$$y = \sqrt{100 - 50} = \sqrt{50} \approx 7.071$$

Step 7 – Second derivative: $A'' < 0 \implies$ maximum **Step 8 – Answer:**

$$\boxed{2x = 2y \approx 14.142 \text{ m; max area } \approx 100 \text{ m}^2}$$

Question 5 – Poster with margins
Step 1 – Variables:

$$w = \text{width of printed area},\ \ h = \text{height of printed area}, wh = 200$$

Step 2 – Solve for h:

$$h = \frac{200}{w}$$

Step 3 – Total dimensions including margins:

$$W = w + 4, \quad H = h + 4$$

Step 4 – Area function:

$$A = (w+4)(h+4) = (w+4)\left(\frac{200}{w} + 4\right) = 4w + \frac{800}{w} + 216$$

Step 5 – Derivative:

$$A' = 4 - \frac{800}{w^2}$$

Step 6 – Critical point:

$$4 - \frac{800}{w^2} = 0 \implies w^2 = 200 \implies w \approx 14.142$$

Step 7 – Solve for h:

$$h = \frac{200}{14.142} \approx 14.142$$

Step 8 – Answer:

$$\boxed{W \approx 18.142 \text{ in},\ \ H \approx 18.142 \text{ in; minimal paper used}}$$

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