Lesson 10: Related Rates

ApexMath

Notes

- Related rates problems involve quantities that change with time.
- A rate of change like $\frac{dr}{dt}$ tells us how fast a variable changes with respect to time.
- When solving, follow these steps:
  1. Identify all **known rates of change** (e.g., $\frac{dr}{dt} = 2$).
  2. Identify all **known variables** at the instant of interest (e.g., $r = 5$).
  3. Identify the **unknown rate** you are solving for (e.g., $\frac{dA}{dt}$).
  4. Write an equation that relates all relevant variables (e.g., $A = \pi r^2$).
  5. Differentiate implicitly with respect to t, remembering the chain rule for variables that depend on t.
  6. Substitute known values and solve for the unknown rate.
- Keep units consistent and always include them in your answer.

Pro Tips

- Draw a diagram for geometric problems (ladders, cones, etc.).
- Wait to plug in numbers until after differentiating.
- Negative rates indicate decreasing quantities.
- Label each variable carefully and write down knowns and unknowns explicitly.

Examples

Example 1: A circle's radius increases at 2 cm/s. Find how fast the area is increasing when $r = 5\,\text{cm}$.

Step 1 – Identify knowns and unknowns:

- Known rate: $\frac{dr}{dt} = 2$
- Known variable: $r = 5$
- Unknown rate: $\frac{dA}{dt}$

Step 2 – Relate the variables:

$$A = \pi r^2$$

Step 3 – Differentiate:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Step 4 – Substitute known values:

$$\frac{dA}{dt} = 2\pi(5)(2) = 20\pi$$

Step 5 – Answer:

$$\boxed{\frac{dA}{dt} = 20\pi \,\mathrm{cm}^2/\mathrm{s}}$$

Example 2: A 10-foot ladder leans against a wall. The bottom slides away at $1\,\mathrm{ft/s}$. How fast is the top sliding down when the bottom is $6\,\mathrm{ft}$ from the wall?

Step 1 – Identify knowns and unknowns:

- Known rate: $\frac{dx}{dt} = 1\,\mathrm{ft/s}$ (bottom moving away)
- Known variable: $x = 6\,\mathrm{ft}$, ladder length $10\,\mathrm{ft}$
- Unknown rate: $\frac{dy}{dt}$ (top sliding down)

Step 2 – Relate the variables:

$$x^2 + y^2 = 10^2$$

Step 3 – Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Step 4 – Solve for $\frac{dy}{dt}$:

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

Step 5 – Substitute values:

$$y = \sqrt{10^2 - 6^2} = 8, \quad \frac{dx}{dt} = 1$$

Step 6 – Answer:

$$\boxed{\frac{dy}{dt} = -\frac{3}{4}\,\text{ft/s}}$$

Example 3: Water is poured into a cone at $20\,\text{cm}^3/\text{s}$. The cone has height $12\,\text{cm}$ and base radius $6\,\text{cm}$. Find $\frac{dh}{dt}$ when $h = 4\,\text{cm}$.

Step 1 – Identify knowns and unknowns:

- Known rate: $\frac{dV}{dt} = 20\,\text{cm}^3/\text{s}$
- Known variable: $h = 4\,\text{cm}$
- Cone dimensions: total height $H = 12\,\text{cm}$, base radius $R = 6\,\text{cm}$
- Unknown rate: $\frac{dh}{dt}$

We want to find how fast the water level rises.

Step 2 – Relate the variables:

- Volume formula: $V = \frac{1}{3}\pi r^2 h$
- Radius r changes with height h.
- Using similar triangles: $\frac{r}{h} = \frac{R}{H} = \frac{6}{12} = \frac{1}{2}$
- Therefore $r = \frac{h}{2}$

Step 3 – Substitute $r = h/2$ into volume:

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now V depends only on h.

Step 4 – Differentiate:

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

Step 5 – Substitute known values:

$$20 = \frac{\pi}{4}(4)^2\frac{dh}{dt}$$

Step 6 – Solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{5}{4\pi} \approx 0.398\,\text{cm/s}$$

Step 7 – Answer:

$$\boxed{\frac{dh}{dt} \approx 0.398\,\text{cm/s}}$$

Common Mistakes

- Forgetting to use the chain rule for variables depending on time.
- Plugging in numbers before differentiating.
- Confusing which quantity is known and which is unknown.
- Ignoring units or inconsistent units.
- Forgetting that a negative rate indicates a decreasing quantity.

Worksheet

1. The radius of a circle increases at $3\,\text{cm/s}$. Find how fast the area increases when $r = 10\,\text{cm}$.
2. A 12-foot ladder leans against a wall. The bottom slides away at $2\,\text{ft/s}$. How fast is the top moving down when the bottom is $5\,\text{ft}$ from the wall?
3. Water is poured into a cone (height $12\,\text{cm}$, radius $6\,\text{cm}$) at $20\,\text{cm}^3/\text{s}$. Find the rate of change of the water's height when $h = 4\,\text{cm}$.
4. Two cars move at right angles: Car A east at $30\,\text{km/h}$, Car B north at $40\,\text{km/h}$. How fast is the distance between them increasing when Car A is 3 km east and Car B is 4 km north of the intersection?
5. **Bonus:** A spherical balloon's radius increases at $0.5\,\text{cm/s}$. How fast is its surface area changing when $r = 8\,\text{cm}$?

Answer Key

1. **Step 1 – Identify knowns/unknowns:** Known: $\frac{dr}{dt} = 3, r = 10$; Unknown: $\frac{dA}{dt}$

   Step 2 – Equation relating variables:

   $$A = \pi r^2$$

   Step 3 – Differentiate:

   $$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

   Step 4 – Substitute values:

   $$\frac{dA}{dt} = 2\pi(10)(3) = 60\pi$$

   Step 5 – Answer:

   $$\boxed{\frac{dA}{dt} = 60\pi\,\text{cm}^2/\text{s}}$$

Step 1 – Identify knowns/unknowns: Known: $\frac{dx}{dt} = 2, x = 5,$ ladder length 12; Unknown: $\frac{dy}{dt}$

Step 2 – Equation relating variables:

$$x^2 + y^2 = 12^2$$

Step 3 – Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Step 4 – Solve for unknown:

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

Step 5 – Substitute values:

$$y = \sqrt{12^2 - 5^2} \approx 10.9, \quad \frac{dx}{dt} = 2$$

Step 6 – Answer:

$$\boxed{\frac{dy}{dt} \approx -0.917\,\text{ft/s}}$$

Step 1 – Identify knowns/unknowns: Known: $\frac{dV}{dt} = 20, h = 4$; Unknown: $\frac{dh}{dt}$

Step 2 – Equation relating variables: Using similar triangles: $r = h/2$

$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{12}h^3$$

Step 3 – Differentiate:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

Step 4 – Substitute values:

$$20 = \frac{\pi}{4}(4)^2\frac{dh}{dt}$$

Step 5 – Solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{5}{4\pi} \approx 0.398$$

Step 6 – Answer:

$$\boxed{\frac{dh}{dt} \approx 0.398\,\text{cm/s}}$$

Step 1 – Identify knowns/unknowns: Known: $x = 3, y = 4$, $\frac{dx}{dt} = 30$, $\frac{dy}{dt} = 40$; Unknown: $\frac{dz}{dt}$

Step 2 – Equation relating variables:

$$x^2 + y^2 = z^2$$

Step 3 – Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

Step 4 – Solve for $\frac{dz}{dt}$:

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \frac{3(30) + 4(40)}{5} = 54$$

Step 5 – Answer:

$$\boxed{\frac{dz}{dt} = 54\,\text{km/h}}$$

Bonus – Identify knowns/unknowns: Known: $r = 8$, $\frac{dr}{dt} = 0.5$; Unknown: $\frac{dA}{dt}$

Step 2 – Equation relating variables:

$$A = 4\pi r^2$$

Step 3 – Differentiate:

$$\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$$

Step 4 – Substitute values:

$$\frac{dA}{dt} = 8\pi(8)(0.5) = 32\pi$$

Step 5 – Answer:

$$\boxed{\frac{dA}{dt} = 32\pi\,\text{cm}^2/\text{s}}$$

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