Riemann Approximation
ApexMath

Notes

What is a Riemann Approximation?

The integral of a function represents the **area under its curve**. But for many functions, computing this area exactly requires tools we will develop later. Riemann approximations give us a way to **estimate** that area right now, using nothing more than rectangles and trapezoids.

The core idea is simple: divide the interval $[a, b]$ into n equal subintervals, build a simple shape on top of each one, compute each shape's area, and add them all up. The result is an approximation of the true area under the curve.

Setting Up the Subintervals:

Given an interval $[a, b]$ divided into n equal pieces, the **width** of each subinterval is:

$$\Delta x = \frac{b-a}{n}$$

The subinterval endpoints are:

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots \quad x_n = b$$

The Four Methods:

1. Left Endpoint Sum (L_n)

Use the *left* endpoint of each subinterval as the height of each rectangle.

$$L_n = \Delta x \left[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right]$$

Notice: we use x_0 through x_{n-1} (we never use the last endpoint x_n).

2. Right Endpoint Sum (R_n)

Use the *right* endpoint of each subinterval as the height of each rectangle.

$$R_n = \Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

Notice: we use x_1 through x_n (we never use the first endpoint x_0).

3. Midpoint Sum (M_n)

Use the *midpoint* of each subinterval as the height of each rectangle.
The midpoint of the i-th subinterval is:

$$m_i = \frac{x_{i-1} + x_i}{2}$$

$$M_n = \Delta x \left[f(m_1) + f(m_2) + \cdots + f(m_n) \right]$$

4. Trapezoidal Sum (T_n)

Instead of rectangles, use *trapezoids*. Each trapezoid has two heights — the function values at the left and right endpoints — and averages them.

$$T_n = \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

Notice: the first and last values are multiplied by 1, and all interior values are multiplied by 2.

Connecting to Riemann Sums:

All four methods are examples of **Riemann sums**. In general, a Riemann sum is written as:

$$\sum_{i=1}^{n} f(x_i^*) \, \Delta x$$

where x_i^* is a chosen sample point in the i-th subinterval. When we let $n \to \infty$, the approximation becomes exact and equals the definite integral:

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x$$

Over- and Underestimates:

- If f is **increasing** on $[a, b]$: L_n underestimates, R_n overestimates.
- If f is **decreasing** on $[a, b]$: L_n overestimates, R_n underestimates.
- T_n overestimates when f is **concave up**, underestimates when f is **concave down**.
- M_n behaves opposite to T_n: it underestimates when f is concave up, overestimates when f is concave down.

Pro Tip

- Always compute $\Delta x = \dfrac{b-a}{n}$ first and write down all the x-values before plugging into any formula. Listing them out prevents mistakes.
- For the trapezoidal rule, a helpful memory trick: the pattern of coefficients is always $1, 2, 2, 2, \ldots, 2, 1$. The endpoints get a 1, everything in the middle gets a 2.
- For midpoint sums, the midpoints are *not* the same as the endpoints. Compute them separately as $m_i = \dfrac{x_{i-1} + x_i}{2}$ for each subinterval.

- More subintervals (n larger) always means a more accurate approximation. As $n \to \infty$, all four methods converge to the exact integral.
- When a problem asks you to write a Riemann sum using sigma notation, identify Δx, your sample point formula x_i^*, and the function f, then substitute into $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$.

Examples

Example 1: Approximate $\displaystyle\int_0^2 x^2\,dx$ using $n = 4$ subintervals with the **Left, Right, and Midpoint** sums.

Step 1: Compute Δx and list all endpoints.

$$\Delta x = \frac{2-0}{4} = \frac{1}{2} = 0.5$$

$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2$$

Step 2: Evaluate $f(x) = x^2$ at each endpoint.

$$f(0) = 0, \quad f(0.5) = 0.25, \quad f(1) = 1, \quad f(1.5) = 2.25, \quad f(2) = 4$$

Step 3: Left Sum L_4 — use x_0 through x_3.

$$L_4 = 0.5\,[f(0) + f(0.5) + f(1) + f(1.5)] = 0.5\,[0 + 0.25 + 1 + 2.25] = 0.5(3.5)$$

$$\boxed{L_4 = 1.75}$$

Step 4: Right Sum R_4 — use x_1 through x_4.

$$R_4 = 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)] = 0.5\,[0.25 + 1 + 2.25 + 4] = 0.5(7.5)$$

$$\boxed{R_4 = 3.75}$$

Step 5: Midpoint Sum M_4 — find the midpoints of each subinterval.

$$m_1 = \frac{0+0.5}{2} = 0.25, \quad m_2 = \frac{0.5+1}{2} = 0.75, \quad m_3 = \frac{1+1.5}{2} = 1.25, \quad m_4 = \frac{1.5+2}{2} = 1.75$$

Evaluate f at each midpoint:

$$f(0.25) = 0.0625, \quad f(0.75) = 0.5625, \quad f(1.25) = 1.5625, \quad f(1.75) = 3.0625$$

$$M_4 = 0.5\,[0.0625 + 0.5625 + 1.5625 + 3.0625] = 0.5(5.25)$$

$$\boxed{M_4 = 2.625}$$

(The exact value is $\int_0^2 x^2\,dx = \frac{8}{3} \approx 2.667$. The midpoint sum is already very close!)

Example 2: Approximate $\displaystyle\int_1^3 \frac{1}{x}\,dx$ using $n = 4$ subintervals with the **Trapezoidal Sum**.

Step 1: Compute Δx and list all endpoints.

$$\Delta x = \frac{3-1}{4} = \frac{2}{4} = 0.5$$

$$x_0 = 1, \quad x_1 = 1.5, \quad x_2 = 2, \quad x_3 = 2.5, \quad x_4 = 3$$

Step 2: Evaluate $f(x) = \dfrac{1}{x}$ at each endpoint.

$$f(1) = 1, \quad f(1.5) \approx 0.6667, \quad f(2) = 0.5, \quad f(2.5) = 0.4, \quad f(3) \approx 0.3333$$

Step 3: Apply the trapezoidal formula. Endpoints get coefficient 1; interior values get coefficient 2.

$$T_4 = \frac{0.5}{2}\,[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)]$$

$$= 0.25\,[1 + 2(0.6667) + 2(0.5) + 2(0.4) + 0.3333]$$

$$= 0.25\,[1 + 1.3334 + 1 + 0.8 + 0.3333]$$

$$= 0.25\,[4.4667]$$

$$\boxed{T_4 \approx 1.1167}$$

(The exact value is $\ln(3) \approx 1.0986$. The trapezoidal approximation is slightly above because $\frac{1}{x}$ is concave up on $[1,3]$.)

Example 3: Express $\displaystyle\int_0^3 x^3\,dx$ as a **Riemann sum using sigma notation** with n subintervals and right endpoints. Then evaluate the limit to find the exact integral.

Step 1: Find Δx and the right endpoint formula.

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

The right endpoint of the i-th subinterval is:

$$x_i = 0 + i \cdot \Delta x = \frac{3i}{n}$$

Step 2: Write the Riemann sum.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(\frac{3i}{n}\right)^3 \cdot \frac{3}{n} = \sum_{i=1}^{n} \frac{27i^3}{n^3} \cdot \frac{3}{n} = \sum_{i=1}^{n} \frac{81i^3}{n^4}$$

Step 3: Factor out the constant and apply the sum formula.

Recall: $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

$$= \frac{81}{n^4} \sum_{i=1}^{n} i^3 = \frac{81}{n^4} \cdot \frac{n^2(n+1)^2}{4} = \frac{81(n+1)^2}{4n^2}$$

Step 4: Take the limit as $n \to \infty$.

$$\int_0^3 x^3\,dx = \lim_{n\to\infty} \frac{81(n+1)^2}{4n^2} = \frac{81}{4} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \frac{81}{4} \cdot 1$$

$$\boxed{\int_0^3 x^3\,dx = \frac{81}{4}}$$

Common Mistakes

- **Using the wrong endpoints for L_n or R_n:** For a left sum, stop at x_{n-1} — do not include x_n. For a right sum, start at x_1 — do not include x_0. A very common error is accidentally including both endpoints in one of these sums.
- **Forgetting to multiply by Δx:** The sum of the function values alone is not the area. You must multiply the entire sum by Δx (or $\frac{\Delta x}{2}$ for the trapezoidal rule).
- **Confusing midpoints with endpoints:** The midpoints m_i are computed from the endpoints but are not the same as the endpoints. Always calculate them separately.
- **Getting the trapezoidal coefficients wrong:** The interior values are multiplied by 2, and the two outer endpoints are multiplied by 1. Do not double the endpoints or forget to double the interior values.

- **Incorrectly applying the $\frac{\Delta x}{2}$ factor in the trapezoidal rule:** The formula divides Δx by 2, not the entire sum. Write $\dfrac{\Delta x}{2}[\ldots]$, not $\Delta x \cdot \dfrac{1}{2}[\ldots]$ applied to just part of the expression.

Worksheet

1. Approximate $\displaystyle\int_0^4 (x^2+1)\,dx$ using $n = 4$ subintervals.
   (a) Left Sum L_4
   (b) Right Sum R_4
   (c) Midpoint Sum M_4

2. Approximate $\displaystyle\int_0^2 e^x\,dx$ using $n = 4$ subintervals.
   (a) Trapezoidal Sum T_4
   (b) Is your answer an overestimate or underestimate? Explain why.

3. Approximate $\int_1^5 \sqrt{x}\,dx$ using $n = 4$ subintervals with the Midpoint Sum M_4.

4. Approximate $\int_0^\pi \sin(x)\,dx$ using $n = 4$ subintervals with the Trapezoidal Sum T_4.

5. For $f(x) = -x^2 + 4$ on $[0, 2]$, use $n = 4$:
   (a) Compute L_4 and R_4.
   (b) Without computing, explain which is an overestimate and which is an underestimate, and why.

6. Express $\int_0^2 x^2\,dx$ as a Riemann sum using **right endpoints** and n subintervals in sigma notation. (Do not evaluate yet.)

7. Using the Riemann sum you set up in Problem 6, evaluate the limit as $n \to \infty$ to find the exact value of $\displaystyle\int_0^2 x^2\,dx$.

$$\text{Recall: } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

8. Express $\displaystyle\int_1^4 (2x+1)\,dx$ as a Riemann sum using right endpoints and n subintervals. Then take the limit to find the exact value.

9. (Challenge) A function $f(x)$ is given by the following table of values:

x	0	1	2	3	4	5
$f(x)$	3	5	4	6	2	7

Use the data in the table to approximate $\displaystyle\int_0^5 f(x)\,dx$ using:

(a) Left Sum L_5

(b) Right Sum R_5

(c) Trapezoidal Sum T_5

10. (Challenge) Show that for any linear function $f(x) = mx + b$ on any interval $[a, b]$, the Trapezoidal Sum T_n gives the **exact** answer regardless of how many subintervals are used.
(Hint: trapezoids fit linear functions perfectly.)

Answer Key

1. $\int_0^4 (x^2+1)\,dx,\ n=4.$

Step 1: $\Delta x = \dfrac{4-0}{4} = 1$. Endpoints: $x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4.$
Step 2: Evaluate $f(x) = x^2 + 1$:

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 5, \quad f(3) = 10, \quad f(4) = 17$$

(a) Left Sum: Use x_0 through x_3.

$$L_4 = 1\,[1 + 2 + 5 + 10] = 18$$

$$\boxed{L_4 = 18}$$

(b) Right Sum: Use x_1 through x_4.

$$R_4 = 1\,[2 + 5 + 10 + 17] = 34$$

$$\boxed{R_4 = 34}$$

(c) Midpoint Sum: Midpoints: $m_1 = 0.5,\ m_2 = 1.5,\ m_3 = 2.5,\ m_4 = 3.5.$

$$f(0.5) = 1.25, \quad f(1.5) = 3.25, \quad f(2.5) = 7.25, \quad f(3.5) = 13.25$$

$$M_4 = 1\,[1.25 + 3.25 + 7.25 + 13.25] = 25$$

$$\boxed{M_4 = 25}$$

(The exact value is $\frac{76}{3} \approx 25.33$. The midpoint sum is excellent.)

2. $\int_0^2 e^x\,dx,\ n=4.$

Step 1: $\Delta x = \dfrac{2-0}{4} = 0.5$. Endpoints: $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2.$
Step 2: Evaluate $f(x) = e^x$:

$$f(0) = 1, \quad f(0.5) \approx 1.6487, \quad f(1) \approx 2.7183, \quad f(1.5) \approx 4.4817, \quad f(2) \approx 7.3891$$

(a) Trapezoidal Sum:

$$T_4 = \frac{0.5}{2}\,[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)]$$

$$= 0.25\,[1 + 2(1.6487) + 2(2.7183) + 2(4.4817) + 7.3891]$$

$$= 0.25\,[1 + 3.2974 + 5.4366 + 8.9634 + 7.3891]$$

$$= 0.25\,[26.0865]$$

$$\boxed{T_4 \approx 6.522}$$

(b) This is an **overestimate**. The function e^x is concave up on $[0, 2]$ (since $f''(x) = e^x > 0$), and the trapezoidal rule always overestimates when the curve is concave up.

3. $\displaystyle\int_1^5 \sqrt{x}\,dx$, $n = 4$, Midpoint Sum.

Step 1: $\Delta x = \dfrac{5-1}{4} = 1$. Endpoints: $1, 2, 3, 4, 5$.
Step 2: Midpoints: $m_1 = 1.5,\ m_2 = 2.5,\ m_3 = 3.5,\ m_4 = 4.5$.
Step 3: Evaluate $f(x) = \sqrt{x}$:

$$f(1.5) \approx 1.2247, \quad f(2.5) \approx 1.5811, \quad f(3.5) \approx 1.8708, \quad f(4.5) \approx 2.1213$$

Step 4:

$$M_4 = 1\,[1.2247 + 1.5811 + 1.8708 + 2.1213]$$

$$\boxed{M_4 \approx 6.798}$$

(The exact value is $\frac{2}{3}(5^{3/2} - 1) \approx 6.787$.)

4. $\displaystyle\int_0^\pi \sin(x)\,dx$, $n = 4$, Trapezoidal Sum.

Step 1: $\Delta x = \dfrac{\pi - 0}{4} = \dfrac{\pi}{4}$. Endpoints: $0,\ \dfrac{\pi}{4},\ \dfrac{\pi}{2},\ \dfrac{3\pi}{4},\ \pi$.
Step 2: Evaluate $f(x) = \sin(x)$:

$$f(0) = 0, \quad f\left(\tfrac{\pi}{4}\right) = \tfrac{\sqrt{2}}{2} \approx 0.7071, \quad f\left(\tfrac{\pi}{2}\right) = 1, \quad f\left(\tfrac{3\pi}{4}\right) \approx 0.7071, \quad f(\pi) = 0$$

Step 3: Apply trapezoidal rule:

$$\begin{aligned} T_4 &= \frac{\pi/4}{2}\,[0 + 2(0.7071) + 2(1) + 2(0.7071) + 0] \\ &= \frac{\pi}{8}\,[0 + 1.4142 + 2 + 1.4142 + 0] \\ &= \frac{\pi}{8}\,[4.8284] \\ &= \frac{4.8284\pi}{8} \approx \frac{15.171}{8} \end{aligned}$$

$$\boxed{T_4 \approx 1.896}$$

(The exact value is 2. Since $\sin(x)$ is concave down on $[0, \pi]$, T_4 slightly underestimates.)

5. $f(x) = -x^2 + 4$ on $[0, 2]$, $n = 4$.

Step 1: $\Delta x = \dfrac{2-0}{4} = 0.5$. Endpoints: $0,\ 0.5,\ 1,\ 1.5,\ 2$.
Step 2: Evaluate $f(x) = -x^2 + 4$:

$$f(0) = 4, \quad f(0.5) = 3.75, \quad f(1) = 3, \quad f(1.5) = 1.75, \quad f(2) = 0$$

(a)

$$L_4 = 0.5\,[f(0) + f(0.5) + f(1) + f(1.5)] = 0.5\,[4 + 3.75 + 3 + 1.75] = 0.5(12.5)$$

$$\boxed{L_4 = 6.25}$$

$$R_4 = 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)] = 0.5\,[3.75 + 3 + 1.75 + 0] = 0.5(8.5)$$

$$\boxed{R_4 = 4.25}$$

(b) The function $f(x) = -x^2 + 4$ is **decreasing** on $[0, 2]$. When a function is decreasing, the left endpoints are always higher than the right endpoints, so the left sum uses taller rectangles than the true curve and **overestimates**. The right sum uses shorter rectangles and **underestimates**.
(Exact value: $\frac{16}{3} \approx 5.333$, which is between 4.25 and 6.25.)

6. Express $\displaystyle\int_0^2 x^2\,dx$ as a Riemann sum using right endpoints.

Step 1: $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$.
Step 2: Right endpoint of the i-th subinterval: $x_i = 0 + i \cdot \dfrac{2}{n} = \dfrac{2i}{n}$.
Step 3: Substitute into the Riemann sum:

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n}$$

$$\boxed{\int_0^2 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4i^2}{n^2} \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^2}{n^3}}$$

7. Evaluate the limit from Problem 6.

Step 1: Factor the constant out of the sum.

$$\sum_{i=1}^{n} \frac{8i^2}{n^3} = \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

Step 2: Apply the sum formula $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

$$= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{8(n+1)(2n+1)}{6n^2} = \frac{4(n+1)(2n+1)}{3n^2}$$

Step 3: Expand and take the limit.

$$\frac{4(2n^2+3n+1)}{3n^2} = \frac{4}{3} \cdot \frac{2n^2+3n+1}{n^2}$$

As $n \to \infty$, the $\frac{3n+1}{n^2}$ terms vanish:

$$\lim_{n\to\infty} \frac{2n^2+3n+1}{n^2} = 2$$

$$\int_0^2 x^2\,dx = \frac{4}{3} \cdot 2$$

$$\boxed{\int_0^2 x^2\,dx = \frac{8}{3}}$$

8. $\int_1^4 (2x+1)\,dx$ using right endpoints with n subintervals.

Step 1: $\Delta x = \frac{4-1}{n} = \frac{3}{n}$.

Step 2: Right endpoint: $x_i = 1 + i \cdot \frac{3}{n} = 1 + \frac{3i}{n}$.

Step 3: Evaluate $f(x_i) = 2\left(1 + \frac{3i}{n}\right) + 1 = 3 + \frac{6i}{n}$.

Step 4: Write the Riemann sum.

$$\sum_{i=1}^{n} \left(3 + \frac{6i}{n}\right) \cdot \frac{3}{n} = \frac{3}{n} \sum_{i=1}^{n} \left(3 + \frac{6i}{n}\right) = \frac{3}{n}\left[3n + \frac{6}{n} \cdot \frac{n(n+1)}{2}\right]$$

Step 5: Simplify.

$$= \frac{3}{n}\left[3n + 3(n+1)\right] = \frac{3}{n}\left[3n + 3n + 3\right] = \frac{3}{n}(6n+3) = 18 + \frac{9}{n}$$

Step 6: Take the limit.

$$\lim_{n\to\infty} \left(18 + \frac{9}{n}\right) = 18$$

$$\boxed{\int_1^4 (2x+1)\,dx = 18}$$

9. (Challenge) Approximating $\int_0^5 f(x)\,dx$ from a table with $\Delta x = 1$.
(a) Left Sum: Use $f(0)$ through $f(4)$.

$$L_5 = 1\,[3 + 5 + 4 + 6 + 2] = 20$$

$$\boxed{L_5 = 20}$$

(b) Right Sum: Use $f(1)$ through $f(5)$.

$$R_5 = 1\,[5 + 4 + 6 + 2 + 7] = 24$$

$$\boxed{R_5 = 24}$$

(c) Trapezoidal Sum:

$$\begin{aligned} T_5 &= \frac{1}{2}\,[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)] \\ &= \frac{1}{2}\,[3 + 2(5) + 2(4) + 2(6) + 2(2) + 7] \\ &= \frac{1}{2}\,[3 + 10 + 8 + 12 + 4 + 7] \\ &= \frac{1}{2}(44) \\ &\boxed{T_5 = 22} \end{aligned}$$

Note: The trapezoidal sum equals $\frac{L_5+R_5}{2} = \frac{20+24}{2} = 22$. *This is always true.*

10. (Challenge) Trapezoidal sum is exact for linear functions.

Let $f(x) = mx + b$ on $[a, b]$ with n subintervals. Then $\Delta x = \dfrac{b-a}{n}$ and:

$$x_k = a + k\,\Delta x$$

The trapezoidal sum is:

$$T_n = \frac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Since f is linear, $f(x_k) = m(a + k\,\Delta x) + b = (ma + b) + mk\,\Delta x$.

Each trapezoid has two parallel sides $f(x_{k-1})$ and $f(x_k)$ and width Δx. The area of a trapezoid is:

$$\text{Area}_k = \frac{\Delta x}{2}\,[f(x_{k-1}) + f(x_k)]$$

For a linear function, the average of the two endpoints $\dfrac{f(x_{k-1}) + f(x_k)}{2}$ equals the exact average value of f over that subinterval (since f has no curvature). Therefore each trapezoid captures the exact area under the line segment, and summing them gives the exact total area regardless of n.

$$T_n = \int_a^b (mx + b)\, dx \quad \text{for all } n \geq 1$$

This confirms the trapezoidal rule is **exact** for any linear function.

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