Separation of Variables
ApexMath

Notes

The Big Idea: Splitting the Equation

In the last lesson you learned what differential equations are and how to verify solutions. Now you will learn how to *find* them.

The most important technique for solving first-order differential equations is called **separation of variables**. The idea is exactly what the name says: rearrange the equation so that everything involving y is on one side and everything involving x is on the other. Then integrate both sides.

A differential equation is **separable** if it can be written in the form:

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

That is, the right-hand side factors into a function of x alone times a function of y alone.

The Four-Step Process

1. **Separate.** Move all y terms (including dy) to the left side and all x terms (including dx) to the right side.

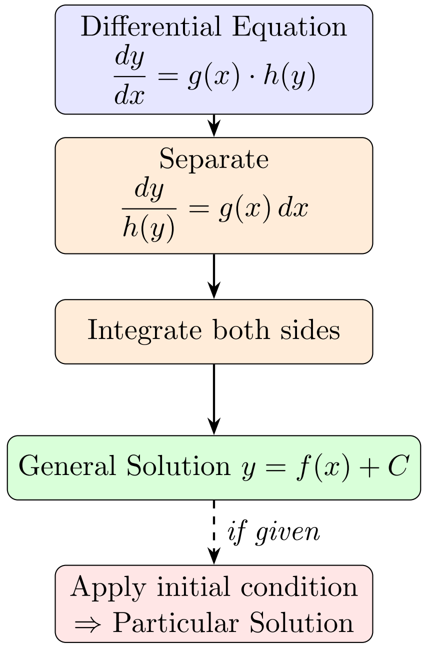
$$\frac{1}{h(y)}\, dy = g(x)\, dx$$

2. **Integrate both sides.**

$$\int \frac{1}{h(y)}\, dy = \int g(x)\, dx$$

3. **Solve for** y (if possible). Write the result explicitly as $y = \ldots$. You only need one constant C — the constants from both integrals combine into one.

4. **Apply the initial condition** (if given) to find the particular solution.



An Important Note About the Constant

When you integrate both sides, each side produces a constant. For example:

$$\int \frac{1}{y}\,dy = \int x\,dx \implies \ln|y| + C_1 = \frac{x^2}{2} + C_2$$

You do not need to keep both constants. Combine them into a single constant: $C = C_2 - C_1$. Write:

$$\ln|y| = \frac{x^2}{2} + C$$

Always write only one C in your final answer.

Dealing with Absolute Values

When you integrate $\frac{1}{y}\,dy$, you get $\ln|y|$. When you solve for y, you exponentiate:

$$|y| = e^{x^2/2+C} = e^C \cdot e^{x^2/2}$$

Since e^C is just a positive constant, we rename it A and drop the absolute value, allowing A to be any nonzero constant:

$$y = Ae^{x^2/2}$$

This is standard practice. You will use this step frequently.

Pro Tip

- **Check separability first.** Before attempting any technique, check whether the right-hand side factors into $g(x) \cdot h(y)$. If it does not factor this way, separation of variables will not work.

- **Do not forget to move dy and dx.** After dividing by $h(y)$, you must also bring dy and dx to the correct sides. Write out every step explicitly — do not try to skip this in your head.

- **Only one constant C is needed.** When you integrate both sides, each produces its own constant. Immediately combine them into a single C on one side.

- **When you see $\ln|y| = \ldots$, exponentiate both sides.** Then write $|y| = e^{\cdots}$, replace e^C with a new constant A, and drop the absolute value by allowing A to be any nonzero real number.

- **Always go back and apply the initial condition last.** Solve the general solution fully before substituting the initial condition. Substituting too early makes the algebra much harder.

Examples

Example 1 (Basic Separation): Solve $\dfrac{dy}{dx} = 3x^2$.

Step 1: Separate. Move dx to the right:

$$dy = 3x^2\, dx$$

Step 2: Integrate both sides.

$$\int dy = \int 3x^2\, dx$$

$$y = x^3 + C$$

$$\boxed{y = x^3 + C}$$

Example 2 (Both Variables Present): Solve $\dfrac{dy}{dx} = xy$.

Step 1: Check separability. The right side $= x \cdot y$, which factors. Separable.
Step 2: Separate. Divide both sides by y and multiply both sides by dx:

$$\frac{1}{y}\, dy = x\, dx$$

Step 3: Integrate both sides.

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

Step 4: Solve for y. Exponentiate both sides:

$$|y| = e^{x^2/2+C} = e^C \cdot e^{x^2/2}$$

Rename e^C as A (any nonzero constant) and drop the absolute value:

$$y = Ae^{x^2/2}$$

$$\boxed{y = Ae^{x^2/2}}$$

Example 3 (With Initial Condition): Solve $\frac{dy}{dx} = 2xy$ with $y(0) = 3$.

Step 1: Separate.

$$\frac{1}{y}\,dy = 2x\,dx$$

Step 2: Integrate both sides.

$$\ln|y| = x^2 + C$$

Step 3: Solve for y.

$$y = Ae^{x^2}$$

Step 4: Apply the initial condition $y(0) = 3$.

$$3 = Ae^0 = A \cdot 1 = A$$

So $A = 3$.

$$\boxed{y = 3e^{x^2}}$$

Example 4 (Fraction on Right Side): Solve $\frac{dy}{dx} = \frac{x}{y}$.

Step 1: Separate. Multiply both sides by y and by dx:

$$y\,dy = x\,dx$$

Step 2: Integrate both sides.

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

Step 3: Solve for y. Multiply both sides by 2:

$$y^2 = x^2 + 2C$$

Rename $2C$ as C (it is still an arbitrary constant):

$$y^2 = x^2 + C$$

Taking the square root:

$$y = \pm\sqrt{x^2 + C}$$

$$\boxed{y = \pm\sqrt{x^2 + C}}$$

Example 5 (Requires Partial Fractions or Careful Algebra): Solve $\dfrac{dy}{dx} = \dfrac{2y}{x}$ with $y(1) = 4$.

Step 1: Separate.

$$\frac{1}{y}\,dy = \frac{2}{x}\,dx$$

Step 2: Integrate both sides.

$$\ln|y| = 2\ln|x| + C$$

Step 3: Solve for y. Use the log property $2\ln|x| = \ln x^2$:

$$\ln|y| = \ln x^2 + C$$

Exponentiate both sides:

$$|y| = e^C \cdot x^2$$

Rename e^C as A:

$$y = Ax^2$$

Step 4: Apply $y(1) = 4$.

$$4 = A(1)^2 = A$$

$$\boxed{y = 4x^2}$$

Common Mistakes

- **Forgetting to divide by $h(y)$ before integrating:** Both sides of the equation must be purely in terms of one variable before you integrate. If y terms remain on the right side, the integration step is invalid.

- **Writing two separate constants instead of one:** After integrating, write C on one side only. Writing C_1 and C_2 and then never combining them is a common source of errors.

- **Dropping the absolute value from $\ln|y|$ too early:** Write $\ln|y| = \ldots$ first. Then exponentiate to get $|y| = e^{\cdots}$. Only after renaming the constant can you drop the absolute value signs.

- **Forgetting $\pm$ when taking a square root:** If the step $y^2 = \ldots$ appears, the solution is $y = \pm\sqrt{\ldots}$. The initial condition will tell you which sign to keep.

- **Applying the initial condition before solving for y:** Always find the general solution in the form $y = \ldots$ first. Then substitute the initial condition to find C. Substituting while y is still implicit makes algebra significantly harder.

Worksheet

1. Solve $\dfrac{dy}{dx} = 5x^4$.

2. Solve $\dfrac{dy}{dx} = \dfrac{x}{y^2}$.

3. Solve $\dfrac{dy}{dx} = -3y$.

4. Solve $\dfrac{dy}{dx} = y^2 x$ with $y(0) = 1$.

5. Solve $\dfrac{dy}{dx} = \dfrac{4x}{y}$ with $y(0) = 2$.

6. Solve $\dfrac{dy}{dx} = e^x y$.

7. Solve $\dfrac{dy}{dx} = \dfrac{3y}{x}$ with $y(2) = 8$.

8. (Challenge) Solve $\dfrac{dy}{dx} = \dfrac{x^2+1}{y+1}$ with $y(0) = 0$.

Answer Key

1. $\dfrac{dy}{dx} = 5x^4$.

Step 1: Separate.

$$dy = 5x^4\,dx$$

Step 2: Integrate both sides.

$$y = x^5 + C$$

$$\boxed{y = x^5 + C}$$

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2. $\dfrac{dy}{dx} = \dfrac{x}{y^2}$.

Step 1: Separate. Multiply both sides by y^2 and by dx:

$$y^2\,dy = x\,dx$$

Step 2: Integrate both sides.

$$\frac{y^3}{3} = \frac{x^2}{2} + C$$

Step 3: Solve for y. Multiply both sides by 3:

$$y^3 = \frac{3x^2}{2} + 3C$$

Rename $3C$ as C:

$$y^3 = \frac{3x^2}{2} + C$$

$$\boxed{y = \sqrt[3]{\frac{3x^2}{2} + C}}$$

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3. $\dfrac{dy}{dx} = -3y$.

Step 1: Separate.

$$\frac{1}{y}\,dy = -3\,dx$$

Step 2: Integrate both sides.

$$\ln|y| = -3x + C$$

Step 3: Solve for y.

$$|y| = e^{-3x+C} = e^C \cdot e^{-3x}$$

Rename e^C as A:

$$\boxed{y = Ae^{-3x}}$$

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4. $\dfrac{dy}{dx} = y^2 x,\ \ y(0) = 1.$

Step 1: Separate. Divide both sides by y^2 and multiply by dx:

$$\frac{1}{y^2}\,dy = x\,dx$$

Step 2: Integrate both sides.

$$\int y^{-2}\,dy = \int x\,dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C$$

Step 3: Solve for y.

$$y = \frac{-1}{\frac{x^2}{2} + C} = \frac{-2}{x^2 + 2C}$$

Rename $2C$ as C:

$$y = \frac{-2}{x^2 + C}$$

Step 4: Apply $y(0) = 1$.

$$1 = \frac{-2}{0 + C} = \frac{-2}{C}$$

$$C = -2$$

$$\boxed{y = \frac{-2}{x^2 - 2} = \frac{2}{2 - x^2}}$$

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5. $\dfrac{dy}{dx} = \dfrac{4x}{y},\ \ y(0) = 2.$

Step 1: Separate.

$$y\,dy = 4x\,dx$$

Step 2: Integrate both sides.

$$\frac{y^2}{2} = 2x^2 + C$$

$$y^2 = 4x^2 + 2C$$

Rename $2C$ as C:

$$y^2 = 4x^2 + C$$

Step 3: Apply $y(0) = 2$.

$$(2)^2 = 4(0)^2 + C \implies 4 = C$$

$$y^2 = 4x^2 + 4$$

Step 4: Solve for y. Since $y(0) = 2 > 0$, take the positive root:

$$\boxed{y = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1}}$$

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6. $\dfrac{dy}{dx} = e^x y$.

Step 1: Separate.

$$\frac{1}{y}\,dy = e^x\,dx$$

Step 2: Integrate both sides.

$$\ln|y| = e^x + C$$

Step 3: Solve for y.

$$|y| = e^{e^x + C} = e^C \cdot e^{e^x}$$

Rename e^C as A:

$$\boxed{y = Ae^{e^x}}$$

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7. $\dfrac{dy}{dx} = \dfrac{3y}{x}$, $y(2) = 8$.

Step 1: Separate.

$$\frac{1}{y}\,dy = \frac{3}{x}\,dx$$

Step 2: Integrate both sides.

$$\ln|y| = 3\ln|x| + C = \ln x^3 + C$$

Step 3: Solve for y.

$$|y| = e^C \cdot x^3$$

Rename e^C as A:

$$y = Ax^3$$

Step 4: Apply $y(2) = 8$.

$$8 = A(2)^3 = 8A \implies A = 1$$

$$\boxed{y = x^3}$$

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8. (Challenge) $\dfrac{dy}{dx} = \dfrac{x^2+1}{y+1}$, $y(0) = 0$.

Step 1: Separate. Multiply both sides by $(y+1)$ and by dx:

$$(y+1)\,dy = (x^2+1)\,dx$$

Step 2: Integrate both sides.

$$\int (y+1)\,dy = \int (x^2+1)\,dx$$

$$\frac{y^2}{2} + y = \frac{x^3}{3} + x + C$$

Step 3: Apply $y(0) = 0$.

$$\frac{0}{2} + 0 = \frac{0}{3} + 0 + C \implies C = 0$$

Step 4: Write the particular solution (left implicit, as solving for y explicitly would require the quadratic formula).

$$\frac{y^2}{2} + y = \frac{x^3}{3} + x$$

Multiply through by 2:

$$\boxed{\frac{y^2}{2} + y = \frac{x^3}{3} + x}$$

Note: It is acceptable to leave this solution in implicit form when solving explicitly for y is not straightforward.

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