# Slope Fields
# ApexMath

## Notes

**The Big Idea: Seeing a Differential Equation**

When you have a differential equation like $\frac{dy}{dx} = x + y$, you cannot always solve it immediately. But you can still *see* what the solutions look like — before doing any algebra.

The key observation is this: at every point $(x, y)$ in the plane, the differential equation tells you the **slope** of the solution curve passing through that point. It is a formula for slope.

A **slope field** (also called a direction field) is a picture made by going to many points in the plane and drawing a tiny line segment at each one with the correct slope. When you step back, the segments form a pattern that reveals the shape of all solution curves at once.

**How to Draw a Slope Field**

To draw a slope field for $\frac{dy}{dx} = f(x, y)$:

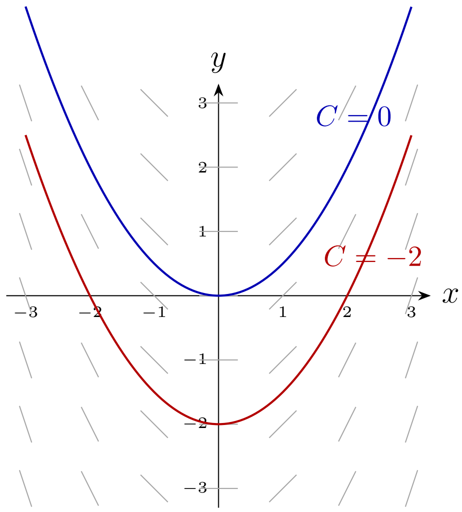
1. Choose a grid of points $(x, y)$.
2. At each point, calculate the slope $m = f(x, y)$.
3. Draw a short line segment through that point with slope $m$.

That is all. No integration required.

**Example:** Draw a slope field for $\frac{dy}{dx} = x$.

At any point $(x, y)$, the slope depends only on $x$, not on $y$. So every point in the same vertical column has the same slope. Let us compute a few:

| $x$ | $y$ | slope $= x$ |
|---|---|---|
| $-2$ | any | $-2$ |
| $-1$ | any | $-1$ |
| $0$ | any | $0$ |
| $1$ | any | $1$ |
| $2$ | any | $2$ |




Notice how the solution curves $y = \dfrac{x^2}{2} + C$ follow the slope field perfectly — they are tangent to every segment they pass through.

**Isoclines**

An **isocline** is a curve along which all slope segments have the same slope. Finding isoclines helps you draw and understand slope fields quickly.

To find the isocline for slope $m$, set $\dfrac{dy}{dx} = m$ and solve for $y$ in terms of $x$.

**Example:** For $\dfrac{dy}{dx} = x - y$, find the isocline where slope $= 0$.

$$x - y = 0 \implies y = x$$

Along the line $y = x$, every slope segment is horizontal (slope $= 0$).

**Sketching Solution Curves Through a Point**

Once you have a slope field, you can sketch the particular solution curve through any given point $(x_0, y_0)$ by:

1. Placing your pencil at $(x_0, y_0)$.
2. Following the direction of the nearby slope segments, staying tangent to them as you move left and right.
3. Never crossing another solution curve (solution curves cannot intersect).

# Pro Tip

- **If $\dfrac{dy}{dx}$ depends only on $x$, all segments in the same vertical column have the same slope.** Look for this pattern first — it makes drawing much faster.

- **If $\frac{dy}{dx}$ depends only on $y$, all segments in the same horizontal row have the same slope.**
- **Where the slope is 0, draw horizontal segments.** These points form the zero isocline. Solution curves have horizontal tangents there.
- **Solution curves never cross each other.** If two curves appeared to cross, a point would have two different slopes, which is impossible. Use this to check your sketch.
- **On the AP exam, you will often be asked to match a slope field to its differential equation.** Check a few key points: what happens along $x = 0$? Along $y = 0$? Where are the horizontal segments? These observations quickly narrow down the options.

# Examples

**Example 1 (Reading a Slope Field):** For $\frac{dy}{dx} = y$, find the slopes at the points $(0, 1)$, $(1, 2)$, and $(2, -1)$.

At any point, the slope equals $y$ (the $y$-coordinate of the point).

- At $(0, 1)$: slope $= 1$
- At $(1, 2)$: slope $= 2$
- At $(2, -1)$: slope $= -1$

When $y > 0$, segments slope upward. When $y < 0$, segments slope downward. When $y = 0$ (the $x$-axis), all segments are horizontal.

**Example 2 (Computing Slopes for a Table):** For $\frac{dy}{dx} = x - y$, complete the slope table and identify the zero isocline.

| $x$ | $y$ | slope $= x - y$ |
|---|---|---|
| 0 | 0 | $0 - 0 = 0$ |
| 1 | 0 | $1 - 0 = 1$ |
| 0 | 1 | $0 - 1 = -1$ |
| 2 | 1 | $2 - 1 = 1$ |
| 1 | 2 | $1 - 2 = -1$ |
| 2 | 2 | $2 - 2 = 0$ |

The slope equals 0 whenever $x = y$. The zero isocline is the line $y = x$. All horizontal segments lie along this line.

**Example 3 (Sketching a Slope Field):** Sketch the slope field for $\frac{dy}{dx} = -\frac{x}{y}$ at the nine points with $x, y \in \{-1, 0, 1\}$.

| $x$ | $y$ | slope $= -x/y$ |
|---|---|---|
| $-1$ | $-1$ | $-1$ |
| $-1$ | $1$ | $1$ |
| $1$ | $-1$ | $1$ |
| $1$ | $1$ | $-1$ |
| $0$ | $-1$ | $0$ |
| $0$ | $1$ | $0$ |
| $-1$ | $0$ | undefined |
| $1$ | $0$ | undefined |
| $0$ | $0$ | undefined |

When $y = 0$, the slope is undefined — draw no segment there. The solution curves for this equation are circles $x^2 + y^2 = C$, which is consistent with the pattern of perpendicular segments.

**Example 4 (Matching a Slope Field to its Equation):** A slope field has horizontal segments along the entire $y$-axis ($x = 0$), and segments become steeper moving away from the $y$-axis. Which equation best fits?

(A) $\frac{dy}{dx} = y$ (B) $\frac{dy}{dx} = x$ (C) $\frac{dy}{dx} = x + y$ (D) $\frac{dy}{dx} = xy$

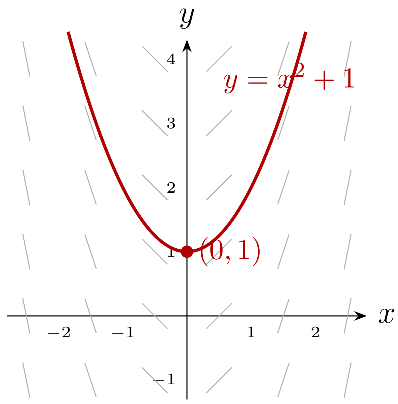
Step 1: Slope is 0 along all of $x = 0$. Test each option at $x = 0$, varying $y$:

- (A): slope $= y$. Is 0 only when $y = 0$, not for all $y$. Eliminated.
- (B): slope $= 0$ for all $y$ when $x = 0$. Possible.
- (C): slope $= 0 + y = y$ at $x = 0$. Is 0 only when $y = 0$. Eliminated.
- (D): slope $= 0 \cdot y = 0$ for all $y$ when $x = 0$. Possible.

Step 2: Segments become steeper only as $x$ changes (same within a column). In (D), slope $= xy$ also depends on $y$, so two points in the same column with different $y$-values would have different slopes. In (B), slope $= x$ depends only on $x$.

**Answer: (B).**

**Example 5 (Sketching a Solution Curve):** Below is the slope field for $\frac{dy}{dx} = 2x$. Sketch the solution curve passing through $(0, 1)$.




The general solution to $\frac{dy}{dx} = 2x$ is $y = x^2 + C$. Applying $y(0) = 1$ gives $C = 1$, so the particular solution is $y = x^2 + 1$. The curve follows the slope segments exactly.

# Common Mistakes

- **Confusing the slope segment with the $y$-value:** The segment shows the *slope* at that point, not the height of the function. A steep upward segment means the function is increasing quickly, not that $y$ is large.

- **Drawing segments that are too long:** Slope segments should be short tick marks. If drawn too long they will overlap and mislead. Keep them small and centered on the grid point.

- **Forgetting that undefined slopes mean vertical tangents:** If the differential equation gives division by zero at a point, draw no segment there or draw a very steep vertical mark. Do not skip the point silently.

- **Drawing solution curves that cross each other:** Distinct solution curves of a well-behaved differential equation never intersect. If your sketch shows curves crossing, re-examine the slope segments and redraw.

- **Ignoring the sign of the slope:** A slope of $-3$ is a steep *downward* segment. Always check whether the computed slope is positive, negative, or zero before drawing.

## Worksheet

1. For $\frac{dy}{dx} = 2x - 1$, compute the slope at each of the following points: $(0, 0)$, $(1, 3)$, $(2, -1)$, $\left(\frac{1}{2}, 5\right)$.

2. For $\frac{dy}{dx} = x + y$, complete the slope table below.

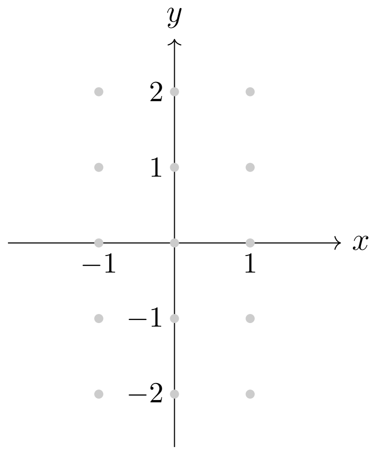
| $x$ | $y$ | slope $= x + y$ |
|---|---|---|
| 0 | 0 | |
| 1 | 0 | |
| 0 | 1 | |
| $-1$ | 1 | |
| 1 | $-1$ | |
| 2 | $-2$ | |

What is the zero isocline for this equation?

3. For $\frac{dy}{dx} = y - x$, which describes the slope segments along $y = x$?

(A) All horizontal. (B) All slope $= 1$. (C) All slope $= -1$. (D) Vary by location.

4. Sketch the slope field for $\frac{dy}{dx} = -y$ on the grid below, using the points with $x \in \{-1, 0, 1\}$ and $y \in \{-2, -1, 0, 1, 2\}$.




5. Using the slope field from problem 4, sketch the solution curve through $(0, 1)$.

6. A slope field has slope $= 0$ along $y = 0$, positive slopes when $y > 0$, and negative slopes when $y < 0$, with steepness increasing as $|y|$ grows. Which equation best matches?

(A) $\frac{dy}{dx} = x$ (B) $\frac{dy}{dx} = y$ (C) $\frac{dy}{dx} = -y$ (D) $\frac{dy}{dx} = x - y$

7. For $\frac{dy}{dx} = \frac{y}{x}$, find the slopes at $(1, 2)$, $(-1, 2)$, $(2, -4)$, and $(3, 3)$. What do the solution curves look like?

8. (Challenge) For the slope field of $\frac{dy}{dx} = x^2 - y$, explain in words what happens to a solution curve that starts where $y > x^2$, one that starts where $y < x^2$, and one that starts on $y = x^2$. What role does $y = x^2$ play?

# Answer Key

**1.** $\dfrac{dy}{dx} = 2x - 1$. Slope depends only on $x$.

At $(0,0)$: slope $= 2(0) - 1 = -1$
At $(1,3)$: slope $= 2(1) - 1 = 1$
At $(2,-1)$: slope $= 2(2) - 1 = 3$
At $\left(\dfrac{1}{2}, 5\right)$: slope $= 2\left(\dfrac{1}{2}\right) - 1 = 0$

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**2.** $\dfrac{dy}{dx} = x + y$.

| $x$ | $y$ | slope |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| $-1$ | 1 | 0 |
| 1 | $-1$ | 0 |
| 2 | $-2$ | 0 |

Zero isocline: set $x + y = 0 \Rightarrow \mathbf{y = -x}$.
Along the line $y = -x$, all slope segments are horizontal.

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**3.** $\dfrac{dy}{dx} = y - x$. Along $y = x$: slope $= x - x = 0$.
All segments are horizontal. **Answer: (A).**

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**4.** Slope field for $\dfrac{dy}{dx} = -y$.

The slope at each point equals $-y$, independent of $x$. All points in the same horizontal row have the same slope:

| $y$ | slope $= -y$ |
|---|---|
| $-2$ | 2 |
| $-1$ | 1 |
| 0 | 0 |
| 1 | $-1$ |
| 2 | $-2$ |

At each grid point, draw a short segment with the slope from the table above.

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**5.** The general solution to $\dfrac{dy}{dx} = -y$ is $y = Ce^{-x}$.
Applying $y(0) = 1$: $1 = Ce^0 = C$, so $C = 1$.
The particular solution is $y = e^{-x}$.
Starting at $(0, 1)$, the curve decreases smoothly toward $y = 0$ as $x$ increases, and rises steeply as $x$ decreases toward negative values.

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**6.** The description matches $\dfrac{dy}{dx} = y$.

- Slope $= 0$ when $y = 0$. ✓
- Slope positive when $y > 0$. ✓
- Slope negative when $y < 0$. ✓
- Steepness grows with $|y|$. ✓

Option (C) $= -y$ has the wrong signs (positive when $y < 0$). Option (A) depends on $x$, not $y$. Option (D) has zero isocline along $y = x$, not $y = 0$.
**Answer: (B).**

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**7.** $\dfrac{dy}{dx} = \dfrac{y}{x}$.

At $(1, 2)$: slope $= \dfrac{2}{1} = 2$
At $(-1, 2)$: slope $= \dfrac{2}{-1} = -2$
At $(2, -4)$: slope $= \dfrac{-4}{2} = -2$
At $(3, 3)$: slope $= \dfrac{3}{3} = 1$

The slope at any point equals $\dfrac{y}{x}$, which is the slope of the straight line from the origin to that point. This means solution curves have the same direction as lines through the origin at every point, so the solution curves *are* lines through the origin: $y = Cx$.

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**8.** (Challenge) Analysis of $\dfrac{dy}{dx} = x^2 - y$.

The slope is positive when $y < x^2$, negative when $y > x^2$, and zero when $y = x^2$.

- **Starts above $y = x^2$:** slope is negative, so the curve decreases, moving back down toward $y = x^2$.
- **Starts below $y = x^2$:** slope is positive, so the curve increases, moving up toward $y = x^2$.
- **Starts on $y = x^2$:** slope is 0 there, so the curve starts horizontally. However, $y = x^2$ is not itself a solution to this DE — the curve will gradually drift away as $x$ changes.

The curve $y = x^2$ acts as an **attractor**: all nearby solution curves are drawn toward it. This is a key qualitative feature of the slope field.

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