Lesson 3: Solving Limits Using Standard Forms

ApexMath

1 Notes on Solving Limits

- To solve limits, we combine algebraic simplification, factoring, and memorized standard limits.
- Key standard limits to memorize (optional for now):

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\quad \lim_{x\to 0}\frac{1-\cos x}{x}=0,\quad \lim_{x\to 0}\frac{\tan x}{x}=1$$

- Strategy for solving:
  1. Check if direct substitution works.
  2. Factor, simplify, or rewrite expressions to match standard forms.
  3. Apply algebraic manipulation for functions like logs or radicals.
  4. Recognize removable discontinuities if present.

2 Deep Walkthrough Examples

Example A. Evaluate $\lim_{x\to 0}\frac{\sin 3x}{x}$

Step 1. Rewrite to match standard form:

$$\frac{\sin 3x}{x}=\frac{\sin 3x}{3x}\cdot 3$$

Step 2. Apply standard limit:

$$\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot 3=1\cdot 3$$

Step 3. Simplify:

$$3$$

$$\boxed{3}$$

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Example B. Evaluate $\lim_{x\to 0}\frac{1-\cos 2x}{x}$

Step 1. Use identity $1-\cos u=2\sin^2(u/2)$:

$$1-\cos 2x=2\sin^2 x$$

Step 2. Split to isolate standard limit:

$$\frac{1-\cos 2x}{x}=2\sin x\cdot\frac{\sin x}{x}$$

Step 3. Apply limits:

$$\lim_{x\to 0} 2 \cdot 0 \cdot 1 = 0$$

$$\boxed{0}$$

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Example C. Evaluate $\lim_{x\to 3} \frac{x^2-9}{x-3}$

Step 1. Factor numerator:

$$x^2 - 9 = (x-3)(x+3)$$

Step 2. Simplify fraction:

$$\frac{(x-3)(x+3)}{x-3} = x+3, \quad x \neq 3$$

Step 3. Substitute $x = 3$:

$$3 + 3 = 6$$

$$\boxed{6}$$

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Example D. Evaluate $\lim_{x\to 0} \frac{\tan 5x}{x}$

Step 1. Rewrite:

$$\frac{\tan 5x}{x} = \frac{\tan 5x}{5x} \cdot 5$$

Step 2. Apply limit:

$$\lim_{x\to 0} 1 \cdot 5 = 5$$

$$\boxed{5}$$

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Example E. Evaluate $\lim_{x\to 0} x \cdot \frac{\sin x}{x}$

Step 1. Split using limit laws:

$$\lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{\sin x}{x}$$

Step 2. Apply limits:

$$0 \cdot 1 = 0$$

$$\boxed{0}$$

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3 Common Mistakes

- Forgetting to adjust the denominator to match the standard limit $(\sin(kx)/(kx))$
- Misapplying the $1-\cos x/x$ formula without factoring or considering constants
- Ignoring removable discontinuities and trying direct substitution too early
- Forgetting to multiply constants outside after rewriting limits

4 Worksheet Problems

1. $\lim_{x\to 0} \frac{\sin 4x}{x}$
2. $\lim_{x\to 0} \frac{1-\cos 3x}{x}$
3. $\lim_{x\to 3} \frac{x^2-9}{x-3}$
4. $\lim_{x\to 0} \frac{\tan 2x}{x}$
5. $\lim_{x\to 0} x \cdot \frac{\sin 5x}{x}$
6. $\lim_{x\to 0} \frac{\sin 2x+\sin 3x}{x}$
7. $\lim_{x\to 0} \frac{1-\cos 5x}{x}$
8. $\lim_{x\to 1} \frac{x^3-1}{x-1}$
9. $\lim_{x\to 0} \frac{\ln(1+x^2)}{\sin 3x}$

5 Answer Key (Step-by-Step)

1. $\lim_{x\to 0} \frac{\sin 4x}{x}$

$$\frac{\sin 4x}{x} = \frac{\sin 4x}{4x} \cdot 4$$

$$\lim_{x\to 0} 1 \cdot 4 = 4$$

$$\boxed{4}$$

2. $\lim_{x\to 0} \frac{1-\cos 3x}{x}$

$$1-\cos 3x = 2\sin^2 \frac{3x}{2}, \quad \frac{1-\cos 3x}{x} = 2\sin \frac{3x}{2} \cdot \frac{\sin \frac{3x}{2}}{x}$$

$$\lim_{x\to 0} 2 \cdot 0 \cdot \frac{3}{2} = 0$$

$\boxed{0}$

3. $\lim_{x\to 3} \frac{x^2-9}{x-3}$

$$x^2 - 9 = (x-3)(x+3), \quad \frac{(x-3)(x+3)}{x-3} = x+3$$

$$x+3\Big|_{x=3} = 6$$

$\boxed{6}$

4. $\lim_{x\to 0} \frac{\tan 2x}{x}$

$$\frac{\tan 2x}{x} = \frac{\tan 2x}{2x} \cdot 2$$

$$\lim_{x\to 0} 1 \cdot 2 = 2$$

$\boxed{2}$

5. $\lim_{x\to 0} x \cdot \frac{\sin 5x}{x}$

$$0 \cdot 5 = 0$$

$\boxed{0}$

6. $\lim_{x\to 0} \frac{\sin 2x + \sin 3x}{x}$

$$\frac{\sin 2x}{x} + \frac{\sin 3x}{x} = \frac{\sin 2x}{2x} \cdot 2 + \frac{\sin 3x}{3x} \cdot 3$$

$$\lim_{x\to 0} 2 + 3 = 5$$

$\boxed{5}$

7. $\lim_{x\to 0} \frac{1-\cos 5x}{x}$

$$1 - \cos 5x = 2\sin^2 \frac{5x}{2}, \quad \frac{2\sin^2 \frac{5x}{2}}{x} = 2\sin \frac{5x}{2} \cdot \frac{\sin \frac{5x}{2}}{x}$$

$$\lim_{x\to 0} 2 \cdot 0 \cdot \frac{5}{2} = 0$$

$\boxed{0}$

8. $\lim_{x\to 1} \frac{x^3-1}{x-1}$

$$x^3 - 1 = (x-1)(x^2+x+1), \quad \frac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1$$

$$x^2+x+1\Big|_{x=1} = 3$$

$\boxed{3}$

9. $\lim_{x\to 0} \frac{\ln(1+x^2)}{\sin 3x}$

Step 1. Factor / manipulate:

$$\frac{\ln(1+x^2)}{\sin 3x} = \frac{\ln(1+x^2)}{x^2} \cdot \frac{x^2}{x} \cdot \frac{1}{\sin 3x} = \frac{\ln(1+x^2)}{x^2} \cdot \frac{x}{\sin 3x}$$

Step 2. Apply limits individually using algebraic simplification (no memorized formula needed):

$$\lim_{x\to 0} \frac{\ln(1+x^2)}{x^2} = 1 \quad \text{(use series expansion or factor)}$$

$$\lim_{x\to 0} \frac{x}{\sin 3x} = \frac{1}{3}$$

Step 3. Multiply results:

$$1 \cdot \frac{1}{3} = \frac{1}{3}$$

$$\boxed{\frac{1}{3}}$$

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