Lesson 7: Tangent Lines and the Instantaneous Slope

ApexMath

Notes on Tangent Lines and the Instantaneous Slope

- A **tangent line** touches a curve at a single point and has the same slope as the curve at that point.
- The **secant line formula** gives the average rate of change between two points:

$$\text{slope of secant} = \frac{f(x+h) - f(x)}{h}$$

- By adding $\lim_{h\to 0}$, we find the instantaneous rate of change (instantaneous slope of the curve at a single point):

$$\text{instantaneous slope at } x = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

- To write the tangent line at a point $(a, f(a))$, we use the **point–slope form**:

$$y - f(a) = m(x - a), \quad \text{where } m = \text{instantaneous slope at } a$$

- Steps to find the tangent line:
  - Use the limit definition to compute the general formula for instantaneous slope.
  - Plug in $x = a$ to get the slope at that point.
  - Use point–slope form with $(a, f(a))$.

Pro Tip

- When using the limit definition to find the instantaneous slope, always make sure that h cancels out in the denominator.
- If h does not cancel, plugging in $h = 0$ will give a division by zero error.
- This is why it's important to expand and simplify the numerator fully before dividing by h.
- Factor carefully if needed to cancel h.
- Always double-check the point you're finding the slope at—substitute correctly after simplifying.
- Remember: the instantaneous slope is the slope of the tangent line, not the secant line.
- For polynomials and other differentiable functions, the derivative will always be simpler after taking the limit.
- **Put it all together now:** simplify first, cancel h, then take the limit to find your slope!

Deep Walkthrough Examples

Example A. Find the tangent line to $f(x) = x^2$ at $x = 1$.

Step 1: Start with the limit definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Substitute $f(x) = x^2$:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Step 3: Expand numerator:

$$(x+h)^2 - x^2 = x^2 + 2xh + h^2 - x^2 = 2xh + h^2$$

Step 4: Divide by h:

$$\frac{2xh + h^2}{h} = 2x + h$$

Step 5: Take the limit as $h \to 0$:

$$f'(x) = 2x$$

So the general instantaneous slope formula is:

$$\boxed{\text{f'(x) = 2x}}$$

Step 6: At $x = 1$:

$$f'(1) = 2(1) = 2$$

$$\boxed{\text{m = 2}}$$

Step 7: Point on curve: $(1, f(1)) = (1, 1)$.

Step 8: Use point–slope form:

$$y - 1 = 2(x - 1)$$

$$\boxed{\text{y - 1 = 2(x - 1)}}$$

—

Example B. Find the tangent line to $f(x) = x^3$ at $x = 2$.

Step 1: Definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2: Substitute $f(x) = x^3$:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

Step 3: Expand:

$$(x+h)^3 - x^3 = x^3 + 3x^2h + 3xh^2 + h^3 - x^3 = 3x^2h + 3xh^2 + h^3$$

Step 4: Divide by h:

$$\frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2$$

Step 5: Limit as $h \to 0$:

$$f'(x) = 3x^2$$

So the general instantaneous slope formula is:

$$\boxed{\text{f'(x) = 3x}^2}$$

Step 6: At $x = 2$:

$$f'(2) = 3(2^2) = 12$$

$$\boxed{\text{m = 12}}$$

Step 7: Point on curve: $(2, f(2)) = (2, 8)$.

Step 8: Equation of tangent line:

$$y - 8 = 12(x - 2)$$

$$\boxed{\text{y - 8 = 12(x - 2)}}$$

—

Common Mistakes

- Forgetting to expand fully before simplifying in the limit definition.
- Canceling h too early without factoring.
- Plugging in $h = 0$ too soon, which makes the fraction undefined.
- Mixing up the average slope (secant line) with the instantaneous slope (tangent line).
- Forgetting to use point–slope form to write the tangent line.

Worksheet Problems

1. Find the tangent line to $f(x) = x^2$ at $x = 3$.
2. Find the general formula for the instantaneous slope of $f(x) = x^2$ using the limit definition.
3. Find the tangent line to $f(x) = x^3$ at $x = 1$.
4. Find the general formula for the instantaneous slope of $f(x) = x^3$ using the limit definition.
5. Find the tangent line to $f(x) = 2x^2 + 3x$ at $x = 1$.

6. Find the general formula for the instantaneous slope of $f(x) = 2x^2 + 3x$ using the limit definition.

—

Detailed Answer Key

1. $f(x) = x^2$ at $x = 3$.

Step 1: Start with the limit definition:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Step 2: Expand numerator:

$$(x+h)^2 - x^2 = 2xh + h^2$$

Step 3: Divide by h:

$$\frac{2xh + h^2}{h} = 2x + h$$

Step 4: Limit as $h \to 0$:

$$f'(x) = 2x$$

Step 5: At $x = 3$:

$$f'(3) = 2(3) = 6$$

$$\boxed{\text{m = 6}}$$

Step 6: Point on curve: $(3, f(3)) = (3, 9)$.

Tangent line:

$$\boxed{\text{y - 9 = 6(x - 3)}}$$

—

2. General instantaneous slope of $f(x) = x^2$.

Work is the same as above, giving:

$$\boxed{\text{f'(x) = 2x}}$$

—

3. $f(x) = x^3$ at $x = 1$.

Step 1: Limit definition:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

Step 2: Expand:

$$(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$$

Step 3: Divide by h:

$$3x^2 + 3xh + h^2$$

Step 4: Limit as $h \to 0$:

$$f'(x) = 3x^2$$

Step 5: At $x = 1$:

$$f'(1) = 3(1^2) = 3$$

$$\boxed{\text{m = 3}}$$

Point on curve: $(1, f(1)) = (1, 1)$.

Tangent line:

$$\boxed{\text{y - 1 = 3(x - 1)}}$$

—

4. General instantaneous slope of $f(x) = x^3$.

From work above:

$$\boxed{\text{f'(x) = 3x}^2}$$

—

5. $f(x) = 2x^2 + 3x$ at $x = 1$.

Step 1: Definition:

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + 3(x+h) - (2x^2 + 3x)}{h}$$

Step 2: Expand numerator:

$$2(x^2 + 2xh + h^2) + 3x + 3h - 2x^2 - 3x = 4xh + 2h^2 + 3h$$

Step 3: Divide by h:

$$\frac{4xh + 2h^2 + 3h}{h} = 4x + 2h + 3$$

Step 4: Limit as $h \to 0$:

$$f'(x) = 4x + 3$$

Step 5: At $x = 1$:

$$f'(1) = 4(1) + 3 = 7$$

$$\boxed{\text{m = 7}}$$

Point on curve: $(1, f(1)) = (1, 5)$.

Tangent line:

$$\boxed{\text{y - 5 = 7(x - 1)}}$$

—

6. General instantaneous slope of $f(x) = 2x^2 + 3x$.

Step 1: Start with the limit definition:

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + 3(x+h) - (2x^2 + 3x)}{h}$$

Step 2: Expand numerator:

$$2(x^2 + 2xh + h^2) + 3x + 3h - 2x^2 - 3x = 4xh + 2h^2 + 3h$$

Step 3: Divide by h:

$$\frac{4xh + 2h^2 + 3h}{h} = 4x + 2h + 3$$

Step 4: Take the limit as $h \to 0$:

$$f'(x) = 4x + 3$$

Put it all together now:

$$\boxed{\text{f'(x) = 4x + 3}}$$

—

Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

End of Lesson • ApexMath
