Lesson 1: The Definition of a Derivative

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Notes

- The slope of a **secant line** through two points on a function $f(x)$ is given by

$$m_{\text{secant}} = \frac{f(x+h) - f(x)}{h}.$$

- This gives the **average rate of change** of the function between x and $x+h$.
- To find the **instantaneous rate of change** (the slope of the tangent line at a single point), we take the limit as $h \to 0$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- This formula is called the **difference quotient**. It is the foundation of derivatives.
- Different notations for derivatives:
  - Lagrange notation: $f'(x)$.
  - Leibniz notation: $\frac{dy}{dx}$, $\frac{df}{dx}$.
  - Leibniz with evaluation: $\left.\frac{dy}{dx}\right|_{x=a}$ means the derivative at $x = a$.
  - Operator notation: $D_x[f(x)]$.

Pro Tips

- Always make sure that h cancels out of the denominator before substituting $h = 0$. Otherwise, you'll get the indeterminate form $\frac{0}{0}$.
- If you see a fraction or a radical, multiply top and bottom by something useful (like the denominator or a conjugate pair) to simplify.
- Remember: a secant line slope becomes the tangent line slope when you let $h \to 0$.
- Different derivative notations may look different, but they all mean the same thing.

Deep Walkthrough Example

Problem: Use the limit definition to find the derivative of $f(x) = x^2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 1: Substitute the function into the formula:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Step 2: Expand the numerator:

$$(x+h)^2 - x^2 = x^2 + 2xh + h^2 - x^2 = 2xh + h^2$$

$$f'(x) = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

Step 3: Factor and simplify:

$$f'(x) = \lim_{h\to 0}(2x + h)$$

Step 4: Plug in $h = 0$:

$$f'(x) = 2x$$

$$\boxed{\text{f'(x) = 2x}}$$

Common Mistakes

- Forgetting to expand correctly when substituting $f(x+h)$.
- Cancelling terms incorrectly (e.g., dropping h too early).
- Plugging in $h = 0$ before simplifying, leading to $\frac{0}{0}$.
- Mixing up secant line slope with tangent line slope.

Worksheet Problems

1. Use the limit definition to find the derivative of $f(x) = x^3$.
2. Use the limit definition to find the derivative of $f(x) = 1/x$.
3. Find the slope of the tangent line to $f(x) = x^2 + 3x$ at $x = 1$.
4. Write the equation of the tangent line to $f(x) = \sqrt{x}$ at $x = 4$.

Detailed Answer Key

1. $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0}(3x^2 + 3xh + h^2)$$

$$= 3x^2$$

$$\boxed{\text{3x}^2}$$

2. $f(x) = 1/x$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
$$= \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x}$$
$$= \lim_{h\to 0} \frac{-h}{h(x+h)x}$$
$$= \lim_{h\to 0} \frac{-1}{(x+h)x}$$
$$= -\frac{1}{x^2}$$

$$\boxed{-\ \frac{1}{x^2}}$$

3. $f(x) = x^2 + 3x$, point $x = 1$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h}$$
$$= \lim_{h\to 0}(2x+h+3)$$
$$= 2x+3$$

At $x = 1$:

$$\left.\frac{dy}{dx}\right|_{x=1} = 2(1) + 3 = 5$$

Tangent line: $y - f(1) = 5(x-1)$. Since $f(1) = 4$: Equation: $y = 5x - 1$.

$$\boxed{\text{y = 5x - 1}}$$

4. $f(x) = \sqrt{x}$, point $x = 4$.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Multiply by conjugate:

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

At $x = 4$:

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{4}$$

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$. Equation: $y = \frac{1}{4}x + 1$.

$$\boxed{\text{y = } \frac{1}{4}x + 1}$$

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