Trigonometric Substitution
ApexMath

Notes

Why Trigonometric Substitution?

Some integrals contain square roots of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$. These cannot be handled by u-substitution alone. The key insight is that the Pythagorean identities eliminate square roots when we replace x with a carefully chosen trig function. For example, if $x = a \sin\theta$, then:

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a\cos\theta$$

The square root disappears entirely.

The Three Cases

Step 1: Identify a from the radical.

Before anything else, look at the expression under the square root and read off the value of a. The number a is always the square root of the constant term.

Pattern in the Radical	**Example**	**Constant Term**	**Value of a**
$\sqrt{a^2 - x^2}$	$\sqrt{9 - x^2}$	$9 = 3^2$	$a = 3$
$\sqrt{a^2 + x^2}$	$\sqrt{x^2 + 25}$	$25 = 5^2$	$a = 5$
$\sqrt{x^2 - a^2}$	$\sqrt{x^2 - 16}$	$16 = 4^2$	$a = 4$

Step 2: Choose the substitution based on the pattern.

Once you know a, choose the substitution from the table below.

Expression	**Substitution**	**Identity Used**	**Simplification**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$	$a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$	$a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	$a\tan\theta$

Domain Restrictions:

- For $x = a\sin\theta$: restrict $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that $\cos\theta \geq 0$.

- For $x = a\tan\theta$: restrict $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so that $\sec\theta > 0$.
- For $x = a\sec\theta$: restrict $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ so that $\tan\theta \geq 0$.

The General Process

Step 1: Identify which case applies and write the substitution $x = \ldots$, $dx = \ldots$.

Step 2: Substitute x, dx, and simplify the square root using the appropriate identity.

Step 3: Integrate the resulting trig expression. This often involves trig identity techniques from the previous lesson.

Step 4: Build a reference triangle from $x = a\sin\theta$ (or $a\tan\theta$ or $a\sec\theta$) and use it to convert all trig expressions back to x.

Building the Reference Triangle

The reference triangle is a right triangle whose sides are labeled according to your substitution. It lets you read off any trig function in terms of x without doing extra algebra.

For example, if $x = 3\sin\theta$, then $\sin\theta = \frac{x}{3}$. Draw a right triangle with:

- Opposite side $= x$
- Hypotenuse $= 3$
- Adjacent side $= \sqrt{9 - x^2}$ (from the Pythagorean theorem)

From this triangle you can read: $\cos\theta = \frac{\sqrt{9-x^2}}{3}$, $\tan\theta = \frac{x}{\sqrt{9-x^2}}$, and so on.

Pro Tip

- The hardest step is identifying which case you have. Look for $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$ under the square root. The sign pattern tells you the substitution.
- Always find dx immediately after choosing your substitution. For $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$. This term is essential and is easy to forget.
- After simplifying the square root, you will almost always be left with a trig integral. Use the techniques from the previous lesson (odd/even powers, identities) to finish it.
- Draw the reference triangle *before* you start back-substituting. Label all three sides, then read off whatever you need directly from the triangle.
- For definite integrals, convert the x-limits to θ-limits using your substitution. This avoids back-substitution entirely and is usually faster.

Examples

Example 1 ($\sqrt{a^2-x^2}$ case): Find $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$.

Step 1: Identify the case and substitute.
We have $\sqrt{9-x^2}=\sqrt{3^2-x^2}$, so $a=3$. Use $x=3\sin\theta$, $dx=3\cos\theta\,d\theta$.
Step 2: Simplify the square root.

$$\sqrt{9-x^2}=\sqrt{9-9\sin^2\theta}=\sqrt{9\cos^2\theta}=3\cos\theta$$

Step 3: Substitute into the integral.

$$\int \frac{1}{3\cos\theta}\cdot 3\cos\theta\,d\theta=\int 1\,d\theta=\theta+C$$

Step 4: Back-substitute. Since $x=3\sin\theta$, we have $\sin\theta=\frac{x}{3}$, so $\theta=\arcsin\left(\frac{x}{3}\right)$.

$$\boxed{\int \frac{1}{\sqrt{9-x^2}}\,dx=\arcsin\left(\frac{x}{3}\right)+C}$$

Example 2 ($\sqrt{a^2+x^2}$ case): Find $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$.

Step 1: We have $\sqrt{x^2+4}=\sqrt{x^2+2^2}$, so $a=2$. Use $x=2\tan\theta$, $dx=2\sec^2\theta\,d\theta$.
Step 2: Simplify the square root.

$$\sqrt{x^2+4}=\sqrt{4\tan^2\theta+4}=\sqrt{4\sec^2\theta}=2\sec\theta$$

Step 3: Substitute.

$$\int \frac{1}{2\sec\theta}\cdot 2\sec^2\theta\,d\theta=\int \sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C$$

Step 4: Build the reference triangle for $x=2\tan\theta$: opposite $=x$, adjacent $=2$, hypotenuse $=\sqrt{x^2+4}$.

$$\sec\theta=\frac{\sqrt{x^2+4}}{2},\qquad \tan\theta=\frac{x}{2}$$

Back-substitute:

$$\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+C=\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right|+C$$

Since $\ln\left|\frac{A}{2}\right|=\ln|A|-\ln 2$ and the $-\ln 2$ absorbs into C:

$$\boxed{\int \frac{1}{\sqrt{x^2+4}}\,dx = \ln\left|x+\sqrt{x^2+4}\right| + C}$$

Example 3 ($\sqrt{x^2-a^2}$ case): Find $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$.

Step 1: We have $\sqrt{x^2-1}$, so $a=1$. Use $x=\sec\theta$, $dx=\sec\theta\tan\theta\,d\theta$.
Step 2: Simplify the square root.

$$\sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=\sqrt{\tan^2\theta}=\tan\theta$$

Step 3: Substitute.

$$\int \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta=\int\tan^2\theta\,d\theta$$

Use the identity $\tan^2\theta=\sec^2\theta-1$:

$$=\int(\sec^2\theta-1)\,d\theta=\tan\theta-\theta+C$$

Step 4: Build the reference triangle for $x=\sec\theta$: hypotenuse $=x$, adjacent $=1$, opposite $=\sqrt{x^2-1}$.

$$\tan\theta=\sqrt{x^2-1},\qquad \theta=\operatorname{arcsec}(x)$$

$$\boxed{\int \frac{\sqrt{x^2-1}}{x}\,dx=\sqrt{x^2-1}-\operatorname{arcsec}(x)+C}$$

Example 4 (Requiring Trig Identities After Substitution): Find $\displaystyle\int\sqrt{4-x^2}\,dx$.

Step 1: $a=2$. Use $x=2\sin\theta$, $dx=2\cos\theta\,d\theta$.
Step 2: Simplify.

$$\sqrt{4-x^2}=\sqrt{4\cos^2\theta}=2\cos\theta$$

Step 3: Substitute.

$$\int 2\cos\theta\cdot 2\cos\theta\,d\theta=4\int\cos^2\theta\,d\theta$$

Apply the power-reducing identity $\cos^2\theta=\dfrac{1+\cos(2\theta)}{2}$:

$$=4\cdot\frac{1}{2}\int(1+\cos(2\theta))\,d\theta=2\left[\theta+\frac{\sin(2\theta)}{2}\right]+C=2\theta+\sin(2\theta)+C$$

Use the double-angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$= 2\theta + 2\sin\theta\cos\theta + C$$

Step 4: Reference triangle for $x = 2\sin\theta$: opposite $= x$, hypotenuse $= 2$, adjacent $= \sqrt{4-x^2}$.

$$\theta = \arcsin\left(\frac{x}{2}\right), \qquad \sin\theta = \frac{x}{2}, \qquad \cos\theta = \frac{\sqrt{4-x^2}}{2}$$

Back-substitute:

$$= 2\arcsin\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$$

$$\boxed{\int \sqrt{4-x^2}\,dx = \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\left(\frac{x}{2}\right) + C}$$

Example 5 (Definite Integral): Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x^2}{\sqrt{x^2+1}}\,dx$.

Step 1: $a = 1$. Use $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$.
Step 2: Simplify.

$$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sec\theta$$

Step 3: Change the limits.

$$x = 0 \implies \tan\theta = 0 \implies \theta = 0$$

$$x = \sqrt{3} \implies \tan\theta = \sqrt{3} \implies \theta = \frac{\pi}{3}$$

Step 4: Substitute.

$$\int_0^{\pi/3} \frac{\tan^2\theta}{\sec\theta}\cdot\sec^2\theta\,d\theta = \int_0^{\pi/3} \tan^2\theta\sec\theta\,d\theta$$

Use $\tan^2\theta = \sec^2\theta - 1$:

$$= \int_0^{\pi/3} (\sec^3\theta - \sec\theta)\,d\theta$$

Recall the standard results:

$$\int \sec^3\theta\,d\theta = \frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2}$$

$$\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$$

So:

$$\int_0^{\pi/3} (\sec^3\theta - \sec\theta)\,d\theta = \left[\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} - \ln|\sec\theta + \tan\theta|\right]_0^{\pi/3}$$

$$= \left[\frac{\sec\theta\tan\theta}{2} - \frac{\ln|\sec\theta + \tan\theta|}{2}\right]_0^{\pi/3}$$

Step 5: Evaluate. At $\theta = \pi/3$: $\sec(\pi/3) = 2$, $\tan(\pi/3) = \sqrt{3}$.

$$\frac{2\cdot\sqrt{3}}{2} - \frac{\ln|2+\sqrt{3}|}{2} = \sqrt{3} - \frac{\ln(2+\sqrt{3})}{2}$$

At $\theta = 0$: $\sec(0) = 1$, $\tan(0) = 0$.

$$\frac{1\cdot 0}{2} - \frac{\ln|1+0|}{2} = 0 - 0 = 0$$

$$\boxed{\int_0^{\sqrt{3}} \frac{x^2}{\sqrt{x^2+1}}\,dx = \sqrt{3} - \frac{\ln(2+\sqrt{3})}{2}}$$

Common Mistakes

- **Choosing the wrong substitution:** The sign pattern under the root determines the case. $a^2 - x^2$ uses sin, $a^2 + x^2$ uses tan, $x^2 - a^2$ uses sec. Mixing these up means the square root will not simplify cleanly.

- **Forgetting to substitute dx:** After choosing $x = a\sin\theta$, you must also replace dx with $a\cos\theta\,d\theta$. Keeping the original dx is a very common error that makes the integral impossible to solve.

- **Forgetting to back-substitute for indefinite integrals:** The final answer must be in terms of x, not θ. Always draw the reference triangle and convert every trig expression back.

- **Not changing the limits for definite integrals:** When converting to a θ-integral, the limits must be recalculated using $\theta = \arcsin(x/a)$, $\arctan(x/a)$, or $\operatorname{arcsec}(x/a)$. Using the original x-limits on the θ-integral is incorrect.

- **Sign errors when simplifying the square root:** $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ only when $\cos\theta \geq 0$, which holds on the standard domain restriction $\theta \in [-\pi/2, \pi/2]$. Outside this range, the absolute value matters.

Worksheet

1. Find $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

2. Find $\displaystyle\int \frac{1}{\sqrt{16+x^2}}\,dx$.

3. Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$.

4. Find $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

   (Use $x = \sin\theta$, then apply the odd-power trig strategy to finish.)

5. Find $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx$.

   (Use $x = 2\tan\theta$*, then use the power-reducing identity.)*

6. Find $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$.

7. Find $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$.

8. Evaluate $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$.

(The answer has a geometric meaning — think about what curve $y = \sqrt{1-x^2}$ represents.)

9. Evaluate $\displaystyle\int_0^2 \frac{1}{(x^2+4)^{3/2}}\,dx$.

10. (Challenge) Find $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$.

    (After substituting $x = 2\tan\theta$, you will need to integrate $\sec^3\theta$ using the reduction formula or IBP.)

11. (Challenge) Find $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$.

(First complete the square: $3 + 2x - x^2 = 4 - (x - 1)^2$. Then use $u = x - 1$ and apply the $\sqrt{a^2 - u^2}$ case.)

Answer Key

1. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

Let $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = 2\cos\theta$.

$$\int \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta = 4\int \sin^2\theta\,d\theta = 4\cdot\frac{1}{2}\int(1-\cos(2\theta))\,d\theta = 2\left[\theta - \frac{\sin(2\theta)}{2}\right] + C$$

$$= 2\theta - \sin(2\theta) + C = 2\theta - 2\sin\theta\cos\theta + C$$

Reference triangle: $\sin\theta = \dfrac{x}{2}$, $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$, $\theta = \arcsin\left(\dfrac{x}{2}\right)$.

$$= 2\arcsin\left(\frac{x}{2}\right) - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} + C$$

$$\boxed{\int \frac{x^2}{\sqrt{4-x^2}}\,dx = 2\arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C}$$

2. $\displaystyle\int \frac{1}{\sqrt{16+x^2}}\,dx$

Let $x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$, $\sqrt{16+x^2} = 4\sec\theta$.

$$\int \frac{4\sec^2\theta}{4\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

Reference triangle: $\tan\theta = \dfrac{x}{4}$, $\sec\theta = \dfrac{\sqrt{16+x^2}}{4}$.

$$\boxed{\int \frac{1}{\sqrt{16+x^2}}\,dx = \ln\left|x + \sqrt{x^2+16}\right| + C}$$

3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$

Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.

$$\int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\cdot 3\tan\theta}\,d\theta = \int \frac{1}{9\sec\theta}\,d\theta = \frac{1}{9}\int\cos\theta\,d\theta = \frac{\sin\theta}{9} + C$$

Reference triangle: $\sec\theta = \dfrac{x}{3}$, $\sin\theta = \dfrac{\sqrt{x^2-9}}{x}$.

$$\boxed{\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \frac{\sqrt{x^2-9}}{9x} + C}$$

4. $\displaystyle\int x^3\sqrt{1-x^2}\,dx$

Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1-x^2} = \cos\theta$.

$$\int \sin^3\theta\cos\theta \cdot \cos\theta\, d\theta = \int \sin^3\theta\cos^2\theta\, d\theta$$

Odd power of sin: save one $\sin\theta$, convert $\sin^2\theta = 1-\cos^2\theta$, let $u = \cos\theta$:

$$= \int (1-u^2)u^2(-du) = -\int (u^2-u^4)\, du = -\frac{u^3}{3} + \frac{u^5}{5} + C$$

Back-substitute $u = \cos\theta = \sqrt{1-x^2}$:

$$= -\frac{(1-x^2)^{3/2}}{3} + \frac{(1-x^2)^{5/2}}{5} + C$$

$$\boxed{\int x^3\sqrt{1-x^2}\, dx = -\frac{(1-x^2)^{3/2}}{3} + \frac{(1-x^2)^{5/2}}{5} + C}$$

5. $\int \frac{1}{(x^2+4)^2}\, dx$

Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $x^2+4 = 4\sec^2\theta$.

$$\int \frac{2\sec^2\theta}{16\sec^4\theta}\, d\theta = \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{8}\cdot\frac{1}{2}\int (1+\cos(2\theta))\, d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C = \frac{1}{16}\left[\theta + \sin\theta\cos\theta\right] + C$$

Reference triangle: $\tan\theta = \frac{x}{2}$, $\sin\theta = \frac{x}{\sqrt{x^2+4}}$, $\cos\theta = \frac{2}{\sqrt{x^2+4}}$, $\theta = \arctan\left(\frac{x}{2}\right)$.

$$= \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}}\right] + C = \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right] + C$$

$$\boxed{\int \frac{1}{(x^2+4)^2}\, dx = \frac{1}{16}\arctan\left(\frac{x}{2}\right) + \frac{x}{8(x^2+4)} + C}$$

6. $\int \frac{\sqrt{x^2-25}}{x}\, dx$

Let $x = 5\sec\theta$, $dx = 5\sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-25} = 5\tan\theta$.

$$\int \frac{5\tan\theta}{5\sec\theta}\cdot 5\sec\theta\tan\theta\, d\theta = 5\int \tan^2\theta\, d\theta = 5\int (\sec^2\theta - 1)\, d\theta = 5(\tan\theta - \theta) + C$$

Reference triangle: $\sec\theta = \frac{x}{5}$, $\tan\theta = \frac{\sqrt{x^2-25}}{5}$, $\theta = \operatorname{arcsec}\left(\frac{x}{5}\right)$.

$$\boxed{\int \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25} - 5\,\text{arcsec}\left(\frac{x}{5}\right) + C}$$

7. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

Let $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$, $\sqrt{9-x^2} = 3\cos\theta$.

$$\int \frac{9\sin^2\theta}{3\cos\theta}\cdot 3\cos\theta\,d\theta = 9\int \sin^2\theta\,d\theta = \frac{9}{2}\int (1-\cos(2\theta))\,d\theta = \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right] + C$$

$$= \frac{9\theta}{2} - \frac{9\sin(2\theta)}{4} + C = \frac{9\theta}{2} - \frac{9\cdot 2\sin\theta\cos\theta}{4} + C = \frac{9\theta}{2} - \frac{9\sin\theta\cos\theta}{2} + C$$

Reference triangle: $\sin\theta = \dfrac{x}{3}$, $\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$, $\theta = \arcsin\left(\dfrac{x}{3}\right)$.

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$

$$\boxed{\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C}$$

8. $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

This is the area of a quarter circle of radius 1 ($y = \sqrt{1-x^2}$ is the upper semicircle). Geometrically:

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$$

We can verify via trig sub. Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$.
Change limits: $x = 0 \Rightarrow \theta = 0$; $\quad x = 1 \Rightarrow \theta = \pi/2$.

$$\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/2} (1+\cos(2\theta))\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

$$\boxed{\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}}$$

9. $\displaystyle\int_0^2 \frac{1}{(x^2+4)^{3/2}}\,dx$

Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$, $(x^2+4)^{3/2} = 8\sec^3\theta$.
Change limits: $x = 0 \Rightarrow \theta = 0$; $\quad x = 2 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \pi/4$.

$$\int_0^{\pi/4} \frac{2\sec^2\theta}{8\sec^3\theta}\,d\theta = \frac{1}{4}\int_0^{\pi/4} \cos\theta\,d\theta = \frac{1}{4}\Big[\sin\theta\Big]_0^{\pi/4} = \frac{1}{4}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{8}$$

$$\boxed{\int_0^2 \frac{1}{(x^2+4)^{3/2}}\,dx = \frac{\sqrt{2}}{8}}$$

10. (Challenge) $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$

Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$, $\sqrt{x^2+4} = 2\sec\theta$.

$$\int \frac{4\tan^2\theta}{2\sec\theta}\cdot 2\sec^2\theta\,d\theta = 4\int \tan^2\theta\sec\theta\,d\theta = 4\int(\sec^3\theta - \sec\theta)\,d\theta$$

Using the standard results $\int\sec^3\theta\,d\theta = \dfrac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|}{2}$ and $\int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|$:

$$= 4\left[\frac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|}{2} - \ln|\sec\theta+\tan\theta|\right] + C$$

$$= 4\left[\frac{\sec\theta\tan\theta}{2} - \frac{\ln|\sec\theta+\tan\theta|}{2}\right] + C = 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| + C$$

Reference triangle: $\tan\theta = \dfrac{x}{2}$, $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$.

$$= 2\cdot\frac{\sqrt{x^2+4}}{2}\cdot\frac{x}{2} - 2\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right| + C = \frac{x\sqrt{x^2+4}}{2} - 2\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| + C$$

Absorbing the $-2\ln(2)$ constant into C:

$$\boxed{\int \frac{x^2}{\sqrt{x^2+4}}\,dx = \frac{x\sqrt{x^2+4}}{2} - 2\ln\left|x+\sqrt{x^2+4}\right| + C}$$

11. (Challenge) $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$

Step 1: Complete the square.

$$3+2x-x^2 = -(x^2-2x-3) = -(x^2-2x+1-4) = 4-(x-1)^2$$

Step 2: Let $u = x-1$, $du = dx$:

$$\int \frac{1}{\sqrt{4-u^2}}\,du$$

Step 3: This is the $\sqrt{a^2-u^2}$ case with $a=2$. Let $u = 2\sin\phi$, $du = 2\cos\phi\,d\phi$, $\sqrt{4-u^2} = 2\cos\phi$:

$$\int \frac{2\cos\phi}{2\cos\phi}\,d\phi = \int 1\,d\phi = \phi + C = \arcsin\left(\frac{u}{2}\right) + C$$

Back-substitute $u = x - 1$:

$$\boxed{\int \frac{1}{\sqrt{3 + 2x - x^2}}\,dx = \arcsin\left(\frac{x-1}{2}\right) + C}$$

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