Lesson 5: Two-Sided Limits and Continuity

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Notes on Two-Sided Limits and Continuity

- A **two-sided limit** $\lim_{x\to a} f(x)$ exists only if both the left-hand limit $\lim_{x\to a^-} f(x)$ and the right-hand limit $\lim_{x\to a^+} f(x)$ exist and are equal.

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \quad \Rightarrow \quad \lim_{x\to a} f(x) \text{ exists.}$$

- If the left-hand limit and right-hand limit are not equal, the two-sided limit does not exist.
- A function is **continuous at** $x = a$ if all three of these conditions are satisfied: 1. $f(a)$ is defined. 2. $\lim_{x\to a} f(x)$ exists. 3. $\lim_{x\to a} f(x) = f(a)$.
- If any of the three conditions fails, the function is **discontinuous** at $x = a$.
- **Types of discontinuities:**
  - **Removable:** Limit exists but $f(a)$ is not defined or does not equal the limit.
  - **Jump:** Left- and right-hand limits exist but are not equal.
  - **Infinite:** The function approaches ∞ or $-\infty$ at $x = a$.
- Always check each of the three continuity conditions step by step.
- Use algebraic simplification, factoring, or piecewise evaluation to compute left- and right-hand limits.

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Deep Walkthrough Examples

Example 1: Does $\lim_{x\to 2} \frac{x^2-4}{x-2}$ exist?

Step 1: Factor numerator.

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}, \quad x \neq 2$$

Step 2: Simplify for $x \neq 2$.

$$f(x) = x + 2$$

Step 3: Find left- and right-hand limits.

$$\lim_{x \to 2^-} f(x) = 4, \quad \lim_{x \to 2^+} f(x) = 4$$

Step 4: Limits are equal, so two-sided limit exists.

$$\boxed{\text{Yes, limit exists and equals 4}}$$

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Example 2: Is $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x \geq 0 \end{cases}$ continuous at $x = 0$?

Step 1: Check $f(0)$ is defined.

$$f(0) = 2 \quad \text{(yes)}$$

Step 2: Evaluate left- and right-hand limits.

$$\lim_{x \to 0^-} f(x) = 1, \quad \lim_{x \to 0^+} f(x) = 2$$

Step 3: Limits are not equal, so two-sided limit does not exist.
Step 4: Since the limit does not exist, $f(x)$ is not continuous at $x = 0$.

$$\boxed{\text{No, function is not continuous at 0}}$$

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Common Mistakes

- Forgetting to check both left-hand and right-hand limits when determining if a two-sided limit exists.
- Confusing existence of a limit with continuity; a function can have a limit at $x = a$ but be discontinuous if $f(a)$ is undefined.
- Not verifying that $\lim_{x \to a} f(x) = f(a)$ for continuity.
- Overlooking piecewise definitions or absolute value functions when computing one-sided limits.
- Assuming a function is continuous everywhere just because it is a simple polynomial; piecewise or rational functions may have discontinuities.

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Worksheet Problems

1. Does $\lim_{x\to 3} \frac{x^2-9}{x-3}$ exist?
2. Is $f(x) = \begin{cases} x^2 & x < 1 \\ 3 & x \geq 1 \end{cases}$ continuous at $x = 1$?
3. Does $\lim_{x\to 0} \frac{|x|}{x}$ exist?
4. Is $f(x) = \frac{1}{x}$ continuous at $x = 0$?
5. Determine if $f(x) = \begin{cases} \sin x & x \neq \pi/2 \\ 1 & x = \pi/2 \end{cases}$ is continuous at $x = \pi/2$.
6. Does $\lim_{x\to 1} \frac{x^3-1}{x-1}$ exist?

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Answer Key

1.

Factor numerator: $\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3}$, $x \neq 3$.
Simplify: $f(x) = x + 3$.

$$\lim_{x\to 3^-} f(x) = 6, \quad \lim_{x\to 3^+} f(x) = 6$$

Two-sided limit exists.

$$\boxed{\text{Yes, limit exists and equals 6}}$$

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2.
Check $f(1)$: defined ($f(1) = 3$).
Left-hand limit: $\lim_{x\to 1^-} x^2 = 1$
Right-hand limit: $\lim_{x\to 1^+} f(x) = 3$
Two-sided limit does not exist $\to$ function is not continuous.

$$\boxed{\text{No, function is not continuous at 1}}$$

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3.

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1, \quad \lim_{x\to 0^+} \frac{|x|}{x} = 1$$

Two-sided limit does not exist.

$$\boxed{\text{No, limit does not exist}}$$

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4.
$f(0)$ is undefined.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty, \quad \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

Two-sided limit does not exist $\to$ function is not continuous.

$$\boxed{\text{No, function is not continuous at } 0}$$

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5.
Check $f(\pi/2) = 1$.

$$\lim_{x \to \pi/2^-} \sin x = 1, \quad \lim_{x \to \pi/2^+} \sin x = 1$$

Two-sided limit exists, and $\lim_{x \to \pi/2} f(x) = f(\pi/2)$.

$$\boxed{\text{Yes, function is continuous at } \pi/2}$$

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6.
Factor numerator: $x^3 - 1 = (x-1)(x^2 + x + 1)$
Simplify: $f(x) = x^2 + x + 1$, $x \neq 1$

$$\lim_{x \to 1^-} f(x) = 3, \quad \lim_{x \to 1^+} f(x) = 3$$

Two-sided limit exists.

$$\boxed{\text{Yes, limit exists and equals } 3}$$

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