U-Substitution
ApexMath

Notes

The Big Idea: Reversing the Chain Rule

When we differentiated composite functions, we used the chain rule:

$$\frac{d}{dx}[F(g(x))] = F'(g(x)) \cdot g'(x)$$

U-substitution is the integration technique that **reverses** this process. If you see an integrand that looks like a function composed with another function, multiplied by the derivative of the inner function, u-substitution will simplify it into a basic integral.

The pattern to look for is:

$$\int F'(g(x)) \cdot g'(x)\, dx$$

If you let $u = g(x)$, then $du = g'(x)\, dx$, and the entire integral collapses to:

$$\int F'(u)\, du$$

which is a much simpler integral.

The Four-Step Process

Step 1: Choose u.

Pick u to be the inner function — the one whose derivative also appears in the integrand. A good choice of u simplifies what remains after substitution.

Step 2: Find du.

Differentiate u with respect to x:

$$\frac{du}{dx} = g'(x) \implies du = g'(x)\, dx$$

Step 3: Substitute.

Replace every x-expression with u and dx with $\frac{du}{g'(x)}$. After a correct substitution, *no x should remain* in the integrand.

Step 4: Integrate, then substitute back.

Integrate in terms of u, then replace u with $g(x)$ to return to the original variable.

Adjusting for a Constant Factor

Sometimes the derivative $g'(x)$ is almost present but differs by a constant factor. For example, if $u = x^2$ then $du = 2x\, dx$, but the integrand might only have $x\, dx$, not $2x\, dx$. In that case, solve for what you need:

$$du = 2x\,dx \implies x\,dx = \frac{du}{2} \implies \frac{1}{2}\,du = x\,dx$$

You can always adjust for a **constant** factor. You **cannot** adjust for a factor that involves x.

U-Substitution for Definite Integrals

When using u-sub on a definite integral $\int_a^b f(g(x)) \cdot g'(x)\,dx$, you have two options:

Option A — Change the limits.
Convert the limits from x-values to u-values using $u = g(x)$:

$$\text{When } x = a: \quad u = g(a) \qquad \text{When } x = b: \quad u = g(b)$$

Then evaluate $\int_{g(a)}^{g(b)} F'(u)\,du$ directly. No back-substitution needed.

Option B — Back-substitute first.
Find the indefinite integral in terms of x, then apply FTC using the original x-limits.
Option A is generally faster and less error-prone.

When U-Substitution Does NOT Work

U-sub requires the derivative of your chosen u to be present (up to a constant) in the integrand. If it is not there, u-sub will not simplify things. For example:

$$\int x \cdot e^{x^2}\,dx \quad \checkmark \quad \text{(works: derivative of } x^2 \text{ is } 2x\text{, and } x \text{ is present)}$$

$$\int e^{x^2}\,dx \quad \times \quad \text{(does not work: no } x \text{ present to match } du = 2x\,dx\text{)}$$

If you choose u and find that the leftover expression still contains x in a way you cannot eliminate, choose a different u or use a different technique.

Pro Tip

- A good first guess for u is usually the expression *inside* a power, inside a square root, inside $e^{(\,\cdot\,)}$, or inside a trig function. Ask yourself: "what would simplify if I replaced this with a single letter?"

- After choosing u, immediately write out du before doing anything else. Then check whether du (or a constant multiple of it) appears in the integrand. If it does not, try a different u.

- When adjusting for a constant, rearrange du to isolate the exact expression in the integrand. For example, if $du = 5x^4\,dx$ but you only have $x^4\,dx$, write $x^4\,dx = \frac{1}{5}\,du$.

- For definite integrals, always convert the limits when you change variables. Forgetting to do this and then plugging the original x-limits into a u-expression is one of the most common errors in this topic.
- After finishing, differentiate your answer to verify it matches the original integrand. This is the most reliable check.

Examples

Example 1 (Basic U-Sub): Find $\int (3x+5)^7 \, dx$.

Step 1: Choose u.
The inner expression is $3x + 5$.

$$u = 3x + 5$$

Step 2: Find du.

$$\frac{du}{dx} = 3 \implies du = 3\,dx \implies dx = \frac{du}{3}$$

Step 3: Substitute.

$$\int (3x+5)^7 \, dx = \int u^7 \cdot \frac{du}{3} = \frac{1}{3} \int u^7 \, du$$

Step 4: Integrate.

$$\frac{1}{3} \cdot \frac{u^8}{8} + C = \frac{u^8}{24} + C$$

Step 5: Substitute back $u = 3x + 5$.

$$\boxed{\int (3x+5)^7 \, dx = \frac{(3x+5)^8}{24} + C}$$

Example 2 (Adjusting for a Constant): Find $\int x^2 \sqrt{x^3+1} \, dx$.

Step 1: Choose u.
The expression inside the square root is $x^3 + 1$.

$$u = x^3 + 1$$

Step 2: Find du.

$$du = 3x^2 \, dx \implies x^2 \, dx = \frac{du}{3}$$

The integrand has $x^2 \, dx$, which matches $\frac{du}{3}$. Good.

Step 3: Substitute.

$$\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\cdot\frac{du}{3} = \frac{1}{3}\int u^{1/2}\,du$$

Step 4: Integrate.

$$\frac{1}{3}\cdot\frac{u^{3/2}}{3/2}+C = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}+C = \frac{2}{9}u^{3/2}+C$$

Step 5: Substitute back.

$$\boxed{\int x^2\sqrt{x^3+1}\,dx = \frac{2}{9}(x^3+1)^{3/2}+C}$$

Example 3 (Trig Function): Find $\displaystyle\int \sin^4(x)\cos(x)\,dx$.

Step 1: Choose u.
Notice that $\cos(x)$ is the derivative of $\sin(x)$. Let:

$$u = \sin(x)$$

Step 2: Find du.

$$du = \cos(x)\,dx$$

The integrand has $\cos(x)\,dx$ already. Perfect.
Step 3: Substitute.

$$\int \sin^4(x)\cos(x)\,dx = \int u^4\,du$$

Step 4: Integrate.

$$\frac{u^5}{5}+C$$

Step 5: Substitute back.

$$\boxed{\int \sin^4(x)\cos(x)\,dx = \frac{\sin^5(x)}{5}+C}$$

Example 4 (Exponential and Logarithmic): Find $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Step 1: Choose u.
The expression in the exponent is $\sqrt{x} = x^{1/2}$.

$$u = \sqrt{x} = x^{1/2}$$

Step 2: Find du.

$$du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx \implies \frac{dx}{\sqrt{x}} = 2\,du$$

Step 3: Substitute.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int e^u \cdot 2\,du = 2\int e^u\,du$$

Step 4: Integrate.

$$2e^u + C$$

Step 5: Substitute back.

$$\boxed{\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C}$$

Example 5 (Definite Integral — Changing the Limits): Evaluate $\displaystyle\int_0^2 x\,e^{x^2}\,dx$.

Step 1: Choose u.

$$u = x^2$$

Step 2: Find du.

$$du = 2x\,dx \implies x\,dx = \frac{du}{2}$$

Step 3: Change the limits.

$$x = 0 \implies u = 0^2 = 0 \qquad x = 2 \implies u = 2^2 = 4$$

Step 4: Substitute everything including the new limits.

$$\int_0^2 x\,e^{x^2}\,dx = \int_0^4 e^u \cdot \frac{du}{2} = \frac{1}{2}\int_0^4 e^u\,du$$

Step 5: Integrate and evaluate.

$$\frac{1}{2}\Big[e^u\Big]_0^4 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1)$$

$$\boxed{\int_0^2 xe^{x^2}\,dx = \frac{e^4 - 1}{2}}$$

Common Mistakes

- **Leaving x-terms in the integral after substituting:** After substituting u and du, check that every x is gone. If any x remains and cannot be written in terms of u, your substitution is wrong.
- **Forgetting to substitute dx:** Students often replace the function with u but forget to replace dx with $\frac{du}{g'(x)}$. The dx must be handled — it carries the adjustment factor.
- **Adjusting by a variable, not a constant:** You can only multiply or divide du by a constant to match the integrand. If the leftover factor involves x, the adjustment is not valid and a different approach is needed.
- **Forgetting to change the limits for definite integrals:** When converting a definite integral, the limits must be updated using $u = g(x)$. Using the original x-values as limits on the u-integral gives a wrong answer.
- **Forgetting to back-substitute for indefinite integrals:** The final answer must be in terms of the original variable x. Leaving the answer in terms of u is incomplete.

Worksheet

1. Find $\displaystyle\int (2x-7)^5\,dx$.

2. Find $\displaystyle\int x(x^2+4)^6\,dx$.

3. Find $\displaystyle\int \cos(5x)\,dx$.

4. Find $\displaystyle\int \sin(x)\cos^3(x)\,dx$.

5. Find $\displaystyle\int \frac{x}{\sqrt{x^2+9}}\,dx$.

6. Find $\displaystyle\int \frac{(\ln x)^3}{x}\,dx$.

7. Find $\displaystyle\int e^x\sqrt{e^x+1}\,dx$.

8. Evaluate $\int_0^1 3x^2(x^3+2)^4\,dx$.

9. Evaluate $\int_0^{\pi/4} \sec^2(x)\tan(x)\,dx$.

10. (Challenge) Find $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$.

Hint: Let $u = 1 - x^2$. After substituting, you will have a leftover x^2 to handle — rewrite $x^2 = 1 - u$.

11. (Challenge) Evaluate $\int_1^4 \frac{\ln x}{x}\,dx$.

Answer Key

1. $\int (2x-7)^5\,dx$

Let $u = 2x - 7$, so $du = 2\,dx \implies dx = \dfrac{du}{2}$.

$$\int u^5 \cdot \frac{du}{2} = \frac{1}{2}\cdot\frac{u^6}{6} + C = \frac{u^6}{12} + C$$

$$\boxed{\int (2x-7)^5\,dx = \frac{(2x-7)^6}{12} + C}$$

Verify: $\dfrac{d}{dx}\left[\dfrac{(2x-7)^6}{12}\right] = \dfrac{6(2x-7)^5\cdot 2}{12} = (2x-7)^5$ ✓

2. $\int x(x^2+4)^6\,dx$

Let $u = x^2 + 4$, so $du = 2x\,dx \implies x\,dx = \dfrac{du}{2}$.

$$\int u^6 \cdot \frac{du}{2} = \frac{1}{2}\cdot\frac{u^7}{7} + C = \frac{u^7}{14} + C$$

$$\boxed{\int x(x^2+4)^6\,dx = \frac{(x^2+4)^7}{14} + C}$$

3. $\int \cos(5x)\,dx$

Let $u = 5x$, so $du = 5\,dx \implies dx = \dfrac{du}{5}$.

$$\int \cos(u)\cdot\frac{du}{5} = \frac{1}{5}\sin(u) + C$$

$$\boxed{\int \cos(5x)\,dx = \frac{1}{5}\sin(5x) + C}$$

4. $\int \sin(x)\cos^3(x)\,dx$

Let $u = \cos(x)$, so $du = -\sin(x)\,dx \implies \sin(x)\,dx = -du$.

$$\int u^3\cdot(-du) = -\frac{u^4}{4} + C$$

$$\boxed{\int \sin(x)\cos^3(x)\,dx = -\frac{\cos^4(x)}{4} + C}$$

5. $\displaystyle\int \frac{x}{\sqrt{x^2+9}}\,dx$

Let $u = x^2+9$, so $du = 2x\,dx \implies x\,dx = \dfrac{du}{2}$.

$$\int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2} \cdot \frac{u^{1/2}}{1/2} + C = u^{1/2} + C$$

$$\boxed{\int \frac{x}{\sqrt{x^2+9}}\,dx = \sqrt{x^2+9} + C}$$

6. $\displaystyle\int \frac{(\ln x)^3}{x}\,dx$

Let $u = \ln x$, so $du = \dfrac{1}{x}\,dx$.

$$\int u^3\,du = \frac{u^4}{4} + C$$

$$\boxed{\int \frac{(\ln x)^3}{x}\,dx = \frac{(\ln x)^4}{4} + C}$$

7. $\displaystyle\int e^x\sqrt{e^x+1}\,dx$

Let $u = e^x + 1$, so $du = e^x\,dx$.

$$\int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C$$

$$\boxed{\int e^x\sqrt{e^x+1}\,dx = \frac{2}{3}(e^x+1)^{3/2} + C}$$

8. $\displaystyle\int_0^1 3x^2(x^3+2)^4\,dx$

Let $u = x^3+2$, so $du = 3x^2\,dx$.
Change the limits:

$$x = 0 \implies u = 0+2 = 2 \qquad x = 1 \implies u = 1+2 = 3$$

$$\int_2^3 u^4\,du = \left[\frac{u^5}{5}\right]_2^3 = \frac{3^5}{5} - \frac{2^5}{5} = \frac{243}{5} - \frac{32}{5} = \frac{211}{5}$$

$$\boxed{\int_0^1 3x^2(x^3+2)^4\,dx = \frac{211}{5}}$$

9. $\displaystyle\int_0^{\pi/4} \sec^2(x)\tan(x)\,dx$

Let $u = \tan(x)$, so $du = \sec^2(x)\,dx$.
Change the limits:

$$x = 0 \implies u = \tan(0) = 0 \qquad x = \frac{\pi}{4} \implies u = \tan\left(\frac{\pi}{4}\right) = 1$$

$$\int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\boxed{\int_0^{\pi/4} \sec^2(x)\tan(x)\,dx = \frac{1}{2}}$$

10. (Challenge) $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

Let $u = 1 - x^2$, so $du = -2x\,dx \implies x\,dx = -\dfrac{du}{2}$.
Also, since $u = 1 - x^2$, we have $x^2 = 1 - u$.
Rewrite the integrand:

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{x^2 \cdot x}{\sqrt{1-x^2}}\,dx$$

Substitute $x^2 = 1 - u$, $x\,dx = -\dfrac{du}{2}$, and $\sqrt{1-x^2} = \sqrt{u}$:

$$= \int \frac{(1-u)}{\sqrt{u}} \cdot \left(-\frac{du}{2}\right) = -\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\,du$$

Split the fraction:

$$= -\frac{1}{2}\int \left(u^{-1/2} - u^{1/2}\right)du = -\frac{1}{2}\left[2u^{1/2} - \frac{2}{3}u^{3/2}\right] + C$$

$$= -u^{1/2} + \frac{1}{3}u^{3/2} + C$$

Substitute back $u = 1 - x^2$:

$$= -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C$$

$$\boxed{\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C}$$

11. (Challenge) $\displaystyle\int_1^4 \frac{\ln x}{x}\,dx$

Let $u = \ln x$, so $du = \dfrac{1}{x}\,dx$.
Change the limits:

$$x = 1 \implies u = \ln(1) = 0 \qquad x = 4 \implies u = \ln(4)$$

$$\int_0^{\ln 4} u\,du = \left[\frac{u^2}{2}\right]_0^{\ln 4} = \frac{(\ln 4)^2}{2} - 0$$

$$\boxed{\int_1^4 \frac{\ln x}{x}\,dx = \frac{(\ln 4)^2}{2}}$$

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