Unit Review
ApexMath

What This Review Covers

This review covers all six topics from the unit. Each section gives you a brief summary of the key formula and strategy, followed by one worked example. Use this sheet to refresh your memory before the test. The worksheet at the end mixes all topics together, just like the real assessment will.

Topic	**Core Formula**
1. Area Between Curves	$A = \int_a^b [f(x) - g(x)]\,dx$
2. Volumes — Disk/Washer	$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx$
3. Volumes — Shell Method	$V = 2\pi \int_a^b x\,f(x)\,dx$
4. Arc Length	$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$
5. Known Cross-Sections	$V = \int_a^b A(x)\,dx$
6. Work	$W = \int_a^b F(x)\,dx$
7. Average Value	$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\,dx$

Topic 1 — Area Between Curves

Key Idea: Subtract the bottom curve from the top curve and integrate over the interval. If the curves are not given explicit bounds, find the intersection points first by setting $f(x) = g(x)$.

$$A = \int_a^b [f(x) - g(x)]\,dx, \qquad f(x) \geq g(x) \text{ on } [a, b]$$

Quick Example: Find the area between $f(x) = x + 2$ and $g(x) = x^2$ from $x = 0$ to $x = 2$.

Test $x = 1$: $f(1) = 3 > g(1) = 1$, so f is on top.

$$A = \int_0^2 [(x+2) - x^2]\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_0^2 = \left(2 + 4 - \frac{8}{3}\right) - 0 = 6 - \frac{8}{3} = \frac{10}{3}$$

Topic 2 — Volumes of Revolution: Disk and Washer Method

Key Idea: Rotating a region around an axis creates a solid whose cross-sections are disks or washers. Always write dV first, identify R and r, then integrate.

$$\textbf{Disk:}\quad V = \pi \int_a^b [f(x)]^2\,dx \qquad \textbf{Washer:}\quad V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right)dx$$

When rotating around a line $y = k$, replace $f(x)$ with $f(x) - k$.

Quick Example: Find the volume when $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is rotated around the x-axis.

Each strip sweeps a disk with radius $R = \sqrt{x}$:

$$V = \pi \int_0^4 (\sqrt{x})^2\,dx = \pi \int_0^4 x\,dx = \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot 8 = 8\pi$$

Topic 3 — Volumes of Revolution: Shell Method

Key Idea: Instead of slicing perpendicular to the axis, wrap thin cylindrical shells around it. Each shell has radius x, height $f(x)$, and thickness dx.

$$V = 2\pi \int_a^b x\,f(x)\,dx$$

The shell method is especially useful when rotating around a *vertical* axis and the integrand would be difficult to set up with washers.

Quick Example: Find the volume when $y = x^2$ from $x = 0$ to $x = 2$ is rotated around the y-axis using the shell method.

Each shell has radius x and height x^2:

$$V = 2\pi \int_0^2 x \cdot x^2\,dx = 2\pi \int_0^2 x^3\,dx = 2\pi \left[\frac{x^4}{4}\right]_0^2 = 2\pi \cdot 4 = 8\pi$$

Topic 4 — Arc Length

Key Idea: Zoom in on a tiny segment of the curve. The Pythagorean theorem gives $ds = \sqrt{(dx)^2 + (dy)^2}$, which after factoring becomes:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Always find $f'(x)$, square it, simplify $1 + [f'(x)]^2$, and look for a perfect square before integrating.

Quick Example: Find the arc length of $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ from $x = 1$ to $x = 2$.

$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$

$$1 + [f'(x)]^2 = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

$$L = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{7}{6} + \frac{1}{4} = \frac{14}{12} + \frac{3}{12} = \frac{17}{12}$$

Topic 5 — Volumes with Known Cross-Sections

Key Idea: A 2D base region sits in the xy-plane. A shape stands perpendicularly at every x-value. Find $s(x) = f(x) - g(x)$, plug into the area formula $A(x)$ for the given shape, then integrate.

$$V = \int_a^b A(x)\,dx$$

Square:	$A(x) = [s(x)]^2$
Semicircle (diameter = s):	$A(x) = \frac{\pi}{8}[s(x)]^2$
Equilateral triangle:	$A(x) = \frac{\sqrt{3}}{4}[s(x)]^2$

Quick Example: Base is $y = \sqrt{x}$ and $y = 0$, $x \in [0, 4]$. Cross-sections are squares. Find the volume.

$s(x) = \sqrt{x}$, so $A(x) = x$.

$$V = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = 8$$

Topic 6 — Work

Key Idea: When force varies with position, slice the displacement into tiny pieces and integrate. The three main applications are springs, cables, and pumping.

$$W = \int_a^b F(x)\,dx$$

- **Springs:** $F(x) = kx$ where x is stretch/compression from natural length.
- **Cables:** $F(x) = \rho(L - x)$ where ρ is weight per unit length.
- **Pumping:** $dW = \rho \cdot x \cdot A(x)\,dx$ where x is depth from the top.

Quick Example: A spring with $k = 50$ N/m is stretched from its natural length to 0.2 m. Find the work.

$$W = \int_0^{0.2} 50x\, dx = 50 \left[\frac{x^2}{2} \right]_0^{0.2} = 25(0.04) = 1 \text{ J}$$

Topic 7 — Average Value of a Function

Key Idea: The average value of f on $[a, b]$ is the height of a rectangle over $[a, b]$ that has the same area as the region under f.

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

The **Mean Value Theorem for Integrals** guarantees a value $c \in [a, b]$ where $f(c) = f_{\text{avg}}$. To find c: compute f_{avg} first, then solve $f(c) = f_{\text{avg}}$.

Quick Example: Find the average value of $f(x) = 2x$ on $[1, 5]$.

$$f_{\text{avg}} = \frac{1}{4} \int_1^5 2x\, dx = \frac{1}{4} \left[x^2\right]_1^5 = \frac{1}{4}(25 - 1) = \frac{24}{4} = 6$$

Worksheet — Mixed Practice

Identify the topic, set up the integral, and solve. Show all steps.

1. Find the area enclosed by $f(x) = 6 - x^2$ and $g(x) = 2$.

2. Find the volume of the solid formed by rotating $y = x^2$ from $x = 0$ to $x = 2$ around the x-axis. (Disk method.)

3. Find the volume of the solid formed by rotating the region between $f(x) = \sqrt{x}$ and $g(x) = x$ from $x = 0$ to $x = 1$ around the y-axis. (Shell method.)

4. Find the arc length of $f(x) = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$.

5. The base of a solid is the region between $y = 2 - x^2$ and $y = 0$ from $x = -\sqrt{2}$ to $x = \sqrt{2}$. Cross-sections perpendicular to the x-axis are semicircles with diameter $= s(x)$. Find the volume.

6. A spring requires 18 J of work to stretch it 0.6 m from its natural length. How much additional work is required to stretch it another 0.4 m (from 0.6 m to 1.0 m)?

7. Find the average value of $f(x) = x^2 + 1$ on $[0, 3]$, then find the value of c in $[0, 3]$ guaranteed by the Mean Value Theorem for Integrals.

8. Find the volume of the solid formed by rotating $y = e^x$ from $x = 0$ to $x = 1$ around the x-axis. (Disk method.)

9. A 12 m cable weighing 6 N/m hangs vertically from a winch. Find the work required to reel in the entire cable.

10. (Challenge) Find the area enclosed by $f(x) = \sin x$ and $g(x) = \cos x$ from $x = \dfrac{\pi}{4}$ to $x = \dfrac{5\pi}{4}$.

Answer Key

1. Area enclosed by $f(x) = 6 - x^2$ and $g(x) = 2$.

Step 1: Find intersection points.

$$6 - x^2 = 2 \implies x^2 = 4 \implies x = \pm 2$$

Step 2: Verify f is on top. Test $x = 0$: $f(0) = 6 > g(0) = 2$. ✓
Step 3: Integrate.

$$A = \int_{-2}^{2} [(6 - x^2) - 2]\, dx = \int_{-2}^{2} (4 - x^2)\, dx$$

The integrand is even, so use symmetry:

$$A = 2\int_{0}^{2} (4 - x^2)\, dx = 2\left[4x - \frac{x^3}{3}\right]_0^2 = 2\left(8 - \frac{8}{3}\right) = 2 \cdot \frac{16}{3}$$

$$\boxed{A = \frac{32}{3}}$$

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2. Volume: $y = x^2$, $x \in [0, 2]$, rotated around x-axis. Disk method.

Step 1: Write dV. Each disk has radius $R = x^2$:

$$dV = \pi (x^2)^2\, dx = \pi x^4\, dx$$

Step 2: Integrate.

$$V = \pi \int_0^2 x^4\, dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \pi \cdot \frac{32}{5}$$

$$\boxed{V = \frac{32\pi}{5}}$$

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3. Volume: region between $f(x) = \sqrt{x}$ and $g(x) = x$, $x \in [0, 1]$, rotated around y-axis. Shell method.

Step 1: The height of each shell is $f(x) - g(x) = \sqrt{x} - x$ (since $\sqrt{x} \geq x$ on $[0, 1]$). The shell radius is x.

$$V = 2\pi \int_0^1 x(\sqrt{x} - x)\, dx = 2\pi \int_0^1 (x^{3/2} - x^2)\, dx$$

Step 2: Integrate.

$$V = 2\pi \left[\frac{x^{5/2}}{5/2} - \frac{x^3}{3}\right]_0^1 = 2\pi \left[\frac{2}{5} - \frac{1}{3}\right] = 2\pi \cdot \frac{6 - 5}{15} = 2\pi \cdot \frac{1}{15}$$

$$\boxed{V = \frac{2\pi}{15}}$$

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4. Arc length of $f(x) = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$.

Step 1: Find $f'(x)$.

$$f'(x) = \frac{2}{3} \cdot \frac{3}{2}x^{1/2} = \sqrt{x}$$

Step 2: Square and add 1.

$$1 + [f'(x)]^2 = 1 + x$$

Step 3: Integrate using $u = 1 + x$, $du = dx$.
When $x = 0$: $u = 1$. When $x = 8$: $u = 9$.

$$L = \int_1^9 \sqrt{u}\, du = \left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{3}(27) - \frac{2}{3}(1) = 18 - \frac{2}{3} = \frac{52}{3}$$

$$\boxed{L = \frac{52}{3}}$$

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5. Cross-sections: base $y = 2 - x^2$ and $y = 0$, $x \in [-\sqrt{2}, \sqrt{2}]$. Semicircle cross-sections, diameter $= s(x)$.

Step 1: $s(x) = 2 - x^2$.
Step 2: Radius $= \dfrac{2 - x^2}{2}$. Area of semicircle:

$$A(x) = \frac{\pi}{2}\left(\frac{2 - x^2}{2}\right)^2 = \frac{\pi(2 - x^2)^2}{8}$$

Step 3: Expand $(2 - x^2)^2 = 4 - 4x^2 + x^4$.
Step 4: Integrate using symmetry (even function on symmetric interval):

$$V = \frac{\pi}{8} \cdot 2\int_0^{\sqrt{2}} (4 - 4x^2 + x^4)\, dx = \frac{\pi}{4}\left[4x - \frac{4x^3}{3} + \frac{x^5}{5}\right]_0^{\sqrt{2}}$$

At $x = \sqrt{2}$: let $s = \sqrt{2}$, so $s^2 = 2$, $s^3 = 2\sqrt{2}$, $s^5 = 4\sqrt{2}$:

$$4\sqrt{2} - \frac{4(2\sqrt{2})}{3} + \frac{4\sqrt{2}}{5} = 4\sqrt{2}\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 4\sqrt{2} \cdot \frac{15 - 10 + 3}{15} = 4\sqrt{2} \cdot \frac{8}{15} = \frac{32\sqrt{2}}{15}$$

$$V = \frac{\pi}{4} \cdot \frac{32\sqrt{2}}{15} = \frac{8\sqrt{2}\,\pi}{15}$$

$$\boxed{V = \frac{8\sqrt{2}\,\pi}{15}}$$

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6. Spring: 18 J to stretch 0.6 m. Find additional work from 0.6 m to 1.0 m.

Step 1: Find k.

$$18 = \int_0^{0.6} kx\,dx = k\left[\frac{x^2}{2}\right]_0^{0.6} = k \cdot \frac{0.36}{2} = 0.18k$$

$$k = \frac{18}{0.18} = 100 \text{ N/m}$$

Step 2: Integrate from 0.6 to 1.0 with $F(x) = 100x$.

$$W = \int_{0.6}^{1.0} 100x\,dx = 100\left[\frac{x^2}{2}\right]_{0.6}^{1.0} = 50\left[(1.0)^2 - (0.6)^2\right] = 50[1 - 0.36] = 50(0.64)$$

$$\boxed{W = 32 \text{ J}}$$

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7. Average value of $f(x) = x^2 + 1$ on $[0, 3]$. Find c.

Step 1: $b - a = 3$.

$$f_{\text{avg}} = \frac{1}{3}\int_0^3 (x^2 + 1)\,dx = \frac{1}{3}\left[\frac{x^3}{3} + x\right]_0^3 = \frac{1}{3}(9 + 3) = \frac{1}{3}(12) = 4$$

Step 2: Solve $f(c) = 4$.

$$c^2 + 1 = 4 \implies c^2 = 3 \implies c = \pm\sqrt{3}$$

Step 3: Keep $c \in [0, 3]$. Discard $c = -\sqrt{3}$.

$$\boxed{f_{\text{avg}} = 4, \quad c = \sqrt{3}}$$

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8. Volume: $y = e^x$, $x \in [0, 1]$, rotated around x-axis. Disk method.

Step 1: Write dV. Radius $R = e^x$:

$$dV = \pi(e^x)^2\,dx = \pi e^{2x}\,dx$$

Step 2: Integrate. Use $\displaystyle\int e^{2x}\,dx = \frac{e^{2x}}{2} + C$:

$$V = \pi\int_0^1 e^{2x}\,dx = \pi\left[\frac{e^{2x}}{2}\right]_0^1 = \frac{\pi}{2}(e^2 - 1)$$

$$\boxed{V = \frac{\pi(e^2 - 1)}{2}}$$

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9. Cable: $L = 12$ m, $\rho = 6$ N/m. Reel in entire cable.

Step 1: At position x (distance already reeled in), remaining hanging length is $(12 - x)$.

$$F(x) = 6(12 - x)$$

Step 2: Integrate from $x = 0$ to $x = 12$.

$$W = \int_0^{12} 6(12 - x)\, dx = 6\left[12x - \frac{x^2}{2}\right]_0^{12} = 6\,[144 - 72] = 6(72)$$

$$\boxed{W = 432 \text{ J}}$$

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10. (Challenge) Area between $f(x) = \sin x$ and $g(x) = \cos x$ from $x = \frac{\pi}{4}$ to $x = \frac{5\pi}{4}$.

Step 1: Determine which function is on top on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Test $x = \frac{\pi}{2}$: $\sin\frac{\pi}{2} = 1 > \cos\frac{\pi}{2} = 0$. So $\sin x \geq \cos x$ on this interval.

Step 2: Set up the integral.

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$$

Step 3: Integrate.

$$A = \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$

Step 4: Evaluate at the bounds.

At $x = \frac{5\pi}{4}$: $-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} = -\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

At $x = \frac{\pi}{4}$: $-\cos\frac{\pi}{4} - \sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

$$A = \sqrt{2} - (-\sqrt{2}) = 2\sqrt{2}$$

$$\boxed{A = 2\sqrt{2}}$$

Thank you for learning with ApexMath!

If you found this lesson helpful, we'd love for you to share your feedback or leave a review.

End of Review • ApexMath
